Synopsis of material from EGA Chapter I, §1.1–1.3

1. Affine schemes

1.1. **Prime spectrum of a ring.** [cf. Liu, 2.1-2.3, Mumford II.1, Eisenbud-Harris I.1]

(1.1.1). Notation: $X = \mathrm{Spec}(A) = \{\text{prime ideals of } A\}$. We write $\mathfrak{j}_x$ for $x \in X$ to emphasize its role as an ideal of A. We have $\mathrm{Spec}(A) = \emptyset$ iff $A = 0$.

$A_x = A_{\mathfrak{j}_x}$ = local ring.

$\mathfrak{m}_x = \mathfrak{j}_x A_x$ = maximal ideal of A_x.

$k(x) = A_x/\mathfrak{m}_x$ = residue field of A_x, equal to the field of fractions of $A/\mathfrak{j}_x$.

$f(x)$ = image in $k(x)$ of $f \in A$, so $f(x) = 0$ iff $f \in \mathfrak{j}(x)$.

$M_x = M \otimes_A A_x$ = localization of an A module M.

$\sqrt{(E)}$ = radical of the ideal generated by $E \subseteq A$.

$V(E) = \{x \in X : E \subseteq \mathfrak{j}_x\} = \{x \in X : f(x) = 0 \text{ for all } f \in E\}$. Then [cf. Liu, 2.1.6]

$$\sqrt{(E)} = \bigcap_{x \in V(E)} \mathfrak{j}_x \tag{1.1.1.1}$$

$V(f) = V(\{f\})$ for $f \in A$.

$D(f) = X \setminus V(f) = \{x \in X : f(x) \neq 0\}$.

Proposition (1.1.2). —

(i) $V(0) = X$, $V(1) = \emptyset$.

(ii) $E \subseteq E' \Rightarrow V(E) \supseteq V(E')$.

(iii) For any collection (E_λ), $V(\bigcup_\lambda E_\lambda) = \bigcap_\lambda V(E_\lambda)$.

(iv) $V(EE') = V(E) \cup V(E')$.

(v) $V(E) = V(\sqrt{(E)})$.

The sets $V(E)$ are the closed subsets of a topology, the *Zariski topology* on X. We understand $X = \mathrm{Spec}(A)$ to have this topology from now on.

(1.1.3). Given $Y \subseteq X$, let $\mathfrak{j}(Y) = \{f \in A : f(x) = 0 \text{ for all } x \in Y\} = \bigcap_{x \in Y} \mathfrak{j}_x$. Then $Y \subseteq Y' \Rightarrow \mathfrak{j}(Y) \supseteq \mathfrak{j}(Y')$ and we have

$$\mathfrak{j}(\bigcup_\lambda Y_\lambda) = \bigcap_\lambda \mathfrak{j}(Y_\lambda) \tag{1.1.3.1}$$

$$\mathfrak{j}(\{x\}) = \mathfrak{j}_x. \tag{1.1.3.2}$$

Proposition (1.1.4). —

(i) For any $E \subseteq A$, $\mathfrak{j}(V(E)) = \sqrt{(E)}$.

(ii) For any $Y \subseteq X$, $V(\mathfrak{j}(Y)) = \overline{Y}$ *is the closure of* Y.

Corollary (1.1.5). — *Closed subsets* $Y \subseteq X$ *and radical ideals* $\mathfrak{a} \subseteq A$ *correspond bijectively via* $Y \mapsto \mathfrak{j}(Y)$, $\mathfrak{a} \mapsto V(\mathfrak{a})$*; the union* $Y_1 \cup Y_2$ *corresponding to* $\mathfrak{j}(Y_1) \cap \mathfrak{j}(Y_2)$ *and an arbitrary intersection* $\bigcap_\lambda Y_\lambda$ *corresponding to* $\sqrt{\sum_\lambda \mathfrak{j}(Y_\lambda)}$.

Corollary (1.1.6). — *If A is a Noetherian ring, then* Spec(A) *is a Noetherian space [the converse does not hold].*

Corollary (1.1.7). — *The closure of $\{x\}$ is the set of $y \in X$ such that $\mathfrak{j}_x \subseteq \mathfrak{j}_y$. Thus $\{x\}$ is closed iff $\mathfrak{j}_x$ is maximal.*

Corollary (1.1.8). — Spec(A) *is a T_0 space.*

(1.1.9). For $f, g \in A$ we have

$$D(fg) = D(f) \cap D(g). \tag{1.1.9.1}$$

We also note that $D(f) = D(g)$ iff $\sqrt{(f)} = \sqrt{(g)}$. In particular this holds if $f = ug$ where $u \in A$ is a unit.

Proposition (1.1.10). — *(i) The sets $D(f)$ for $f \in A$ form a base of open sets on X.*
(ii) $D(f)$ is quasi-compact, and in particular, so is $X = D(1)$.

Proposition (1.1.11). — $\mathrm{Spec}(A/\mathfrak{a})$ *is canonically identified with the closed subset* $V(\mathfrak{a}) \subseteq \mathrm{Spec}(A)$.

Corollary (1.1.12). — $\mathrm{Spec}(A)$ *and* $\mathrm{Spec}(A/\sqrt{(0)})$ *are canonically homeomorphic.*

Proposition (1.1.13). — $\mathrm{Spec}(A)$ *is an irreducible space iff $A/\sqrt{(0)}$ is a domain, i.e., $\sqrt{(0)}$ is prime.*

Corollary (1.1.14). — *(i) In the correspondence between closed subsets of X and radical ideals of A, the irreducible closed subsets correspond to the prime ideals. In particular, the irreducible components of X correspond to the minimal primes.*
(ii) $x \mapsto \overline{\{x\}}$ is a bijection from points of X to irreducible closed subsets of X, i.e., each irreducible closed subset has a unique generic point.

Proposition (1.1.15). — *If $\mathcal{I}$ is an ideal contained in the Jacobson radical $\mathfrak{R}(A)$, the whole space X is the unique open neighborhood of $V(\mathcal{I})$.*

1.2. **Functorial properties of the prime spectrum of a ring.**

(1.2.1). A ring homomorphism $\phi\colon A' \to A$ induces a map

$${}^a\phi\colon X = \mathrm{Spec}(A) \to X' = \mathrm{Spec}(A')$$

by ${}^a\phi(x) = \phi^{-1}(\mathfrak{j}_x)$. We denote by ϕ^x the injective homomorphism of integral domains $A'/\phi^{-1}(\mathfrak{j}_x) \to A/\mathfrak{j}_x$ or its extension to their fraction fields $\phi^x\colon k({}^a\phi(x)) \to k(x)$. For any $f' \in A'$, we then have

$$\phi^x(f'({}^a\phi(x))) = (\phi(f'))(x). \tag{1.2.1.1}$$

Proposition (1.2.2). — *(i) For any $E' \subseteq A'$, we have*

$${}^a\phi^{-1}(V(E')) = V(\phi(E')), \tag{1.2.2.1}$$

and in particular, for any $f' \in A$,

$${}^a\phi^{-1}(D(f')) = D(\phi(f')). \tag{1.2.2.2}$$

(ii) For any ideal $\mathfrak{a} \subseteq A$*, we have*

$$\overline{{}^a\phi(V(\mathfrak{a}))} = V(\phi^{-1}(\mathfrak{a})). \tag{1.2.2.3}$$

Corollary (1.2.3). — *${}^a\phi$ is continuous.*

In fact, Spec is a contravariant functor from commutative rings to topological spaces.

Corollary (1.2.4). — *[cf. Liu, 2.1.7 (b)] Suppose that ϕ is surjective, or more generally, that every $f \in A$ has the form $f = h\phi(f')$ for some $f' \in A'$ and invertible $h \in A$. Then ${}^a\phi$ is a homeomorphism from X onto ${}^a\phi(X)$.*

(1.2.5). In particular, when ϕ is the canonical homomorphism $A \to A/\mathfrak{a}$, then ${}^a\phi$ is the inclusion of $\mathrm{Spec}(A/\mathfrak{a})$, identified with $V(\mathfrak{a})$, into $\mathrm{Spec}(A)$.

Corollary (1.2.6). — *[cf. Liu, 2.1.7 (c)] For any multiplicative set $S \subseteq A$,* $\mathrm{Spec}(S^{-1}A)$ *is canonically homeomorphic to the subspace $\{x \in X : \mathfrak{j}_x \cap S = \emptyset\}$ of $X = \mathrm{Spec}(A)$.*

Corollary (1.2.7). — *${}^a\phi(X)$ is dense in X' iff* $\ker(\phi)$ *consists of nilpotent elements.*

1.3. **Sheaf associated to a module.** [cf. Liu, Section 5.1.2]

(1.3.1). Let M be an A module, $f \in A$, $S_f = \{1, f, f^2, \ldots\}$, $A_f = S_f^{-1}A$, $M_f = S_f^{-1}M$. Let $S'_f = \{g \in A : g \text{ divides } f^n \text{ for some } n\}$ be the saturation of S_f, so $S_f'^{-1}A = S_f^{-1}A$, $S_f'^{-1}M = S_f^{-1}M$ canonically (0, 1.4.3).

Lemma (1.3.2). — *The following are equivalent: (a) $g \in S'_f$, (b) $S'_g \subseteq S'_f$, (c) $f \in \sqrt{(g)}$, (d) $\sqrt{(f)} \subseteq \sqrt{(g)}$, (e) $V(g) \subseteq V(f)$, (f) $D(f) \subseteq D(g)$.*

(1.3.3). If $D(f) = D(g)$ it follows that $M_f = M_g$, and for $D(f) \supseteq D(g)$ there is a canonical homomorphism, functorial in M,

$$\rho_{g,f} \colon M_f \to M_g,$$

satisfying

$$\rho_{h,g} \circ \rho_{g,f} \tag{1.3.3.1}$$

for $D(f) \supseteq D(g) \supseteq D(h)$. Given $x \in \mathrm{Spec}(A)$, the localization M_x is the direct limit $\varinjlim M_f$ of the system formed by the modules M_f and homomorphisms $\rho_{g,f}$, as f varies over $A \setminus \mathfrak{j}_f$. Write

$$\rho^f_x \colon M_f \to M_x$$

for the canonical homomorphism, for $f \in A \setminus \mathfrak{j}_x$ (*i.e.*, for $x \in D(f)$).

Definition (1.3.4). — The *structure sheaf* of $X = \mathrm{Spec}(A)$ (resp. *sheaf associated to an A module M*), denoted $\widetilde{A}$ or $\mathcal{O}_X$ (resp. $\widetilde{M}$) is the sheaf of rings (resp. sheaf of $\mathcal{O}_X$ modules) associated to the presheaf $D(f) \mapsto A_f$ (resp. $D(f) \mapsto M_f$) on the base $\mathcal{B}$ of open sets of the form $D(f)$ (see (1.1.10) and (0, 3.2.1 and 3.5.6)).

By (0, 3.2.1), the stalks $\widetilde{A}_x$ (resp. $\widetilde{M_x}$) are just A_x (resp. M_x), and we have canonical homomorphisms $\theta_f\colon A_f \to \Gamma(D(f), \widetilde{A})$ and similarly for M, such that

$$(\theta_f(m))_x = \rho_x^f(m) \tag{1.3.4.1}$$

for all $m \in M_f$.

Proposition (1.3.5). — *[cf. Liu, 5.1.5 (b)]* $M \mapsto \widetilde{M}$ *is an exact, contravariant functor from* A *modules to sheaves of* $\widetilde{A}$ *modules.*

Proposition (1.3.6). — *[cf. Liu, 2.3.7] For every* $f \in A$*, the open set* $D(f)$ *is canonically homeomorphic to* $\operatorname{Spec}(A_f)$*, and the sheaf* $\widetilde{M_f}$ *associated to the* A_f *module* M_f *coincides under this identification with* $\widetilde{M}|D(f)$*.*

Theorem (1.3.7). — *[cf. Liu, 2.3.1 (a)] The canonical homomorphism* $\theta_f\colon M_f \to \Gamma(D(f), \widetilde{M})$ *is an isomorphism. In particular,* $M \cong \Gamma(X, \widetilde{M})$*.*

Corollary (1.3.8). — *Given two* A *modules* M*,* N*, the functorial map* $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ *is bijective, i.e., the functor* $M \mapsto \widetilde{M}$ *is fully faithful. In particular* $\widetilde{M} = 0$ *iff* $M = 0$*.*

Corollary (1.3.9). — *(i) Given* $u\colon M \to N$*, the sheaves associated to* $\ker u$*,* $\operatorname{im} u$*,* $\operatorname{coker} u$ *are* $\ker \widetilde{u}$*,* $\operatorname{im} \widetilde{u}$*,* $\operatorname{coker} \widetilde{u}$*. Thus* $\widetilde{u}$ *is injective (resp. surjective, bijective) iff* u *is.*

(ii) The functor $M \to \widetilde{M}$ *commutes with all inductive limits, in particular with all direct sums [Liu, 5.1.5 (a)].*

Note that the sheaves associated to A modules form an Abelian category, by (1.3.8). If M is finitely generated, then there is a surjection $\widetilde{A}^n \to \widetilde{M}$, so $\widetilde{M}$ is generated by a finite family of global sections.

(1.3.10). If $N \subseteq M$ is a submodule, then the inclusion induces an injective homomorphism $\widetilde{N} \hookrightarrow \widetilde{M}$. Hence we can and will identify $\widetilde{N}$ with a subsheaf of $\widetilde{M}$. With this identification, (1.3.9) implies that the functor $M \to \widetilde{M}$ preserves sums and finite intersections of submodules.

Corollary (1.3.11). — *On the category of sheaves associated to* A *modules, the global section functor* Γ *is exact.*

Corollary (1.3.12). — *(i)* $M \mapsto \widetilde{M}$ *commutes with tensor products [Liu, 5.1.5 (d)].*

(ii) For finitely presented M*, the sheaf associated to* $\operatorname{Hom}_A(M, N)$ *is canonically identified with* $\mathcal{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$*.*

(1.3.13). If B is an A algebra, then $\widetilde{B}$ is a sheaf of $\widetilde{A}$ algebras, and if M is a B module, then $\widetilde{M}$ is a sheaf of $\widetilde{B}$ modules, of finite type iff M is a finitely generated B module. The results in (1.3.8–1.3.12) apply in this setting as well. If B (resp. M) is graded, then so is $\widetilde{B}$ (resp. $\widetilde{M}$) [see (0, 4.1.4)].

(1.3.14). If B is an A algebra, $M \subseteq B$ an A submodule, and $C \subseteq B$ the subalgebra generated by M, then $\widetilde{C}$ is the $\widetilde{A}$ subalgebra of $\widetilde{B}$ generated by $\widetilde{M}$ [cf. (0, 4.1.3)].