Math 252 Fall 2012 Homework Problems

Lecture 1

1. The dihedral group D_{2n} of order $2n$ acts naturally on $\mathbb{R}^2$ by reflections and rotations. Find the matrices of two generating reflections s, t in the corresponding matrix representation, and verify by computation the relation $sts\cdots = tst\cdots$ (n factors on each side). Hint: regard this as a complex matrix representation, and diagonalize the matrix of st.

2. The symmetric group S_3 is isomorphic to the dihedral group D_6. Identifying these two groups, construct an isomorphism between the natural representation of D_6 on $\mathbb{R}^2$, and the 2-dimensional real representation of S_3 on V/U, where $V = \mathbb{R}^3$ is the standard representation, and U is the submodule spanned by the invariant vector $(1, 1, 1)$.

3. The *quaternion group* is the 8-element subgroup $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ of the multiplicative group of non-zero quaternions. Find a faithful 2-dimensional complex matrix representation of Q. Hint: if you define scalar multiplication by $\mathbb{C}$ on the quaternions $\mathbb{H}$ to be right multiplication, then left multiplication by any $x \in \mathbb{H}$ is $\mathbb{C}$-linear.

4. Are the quaternion group Q and the dihedral group D_8 isomorphic?

Lecture 2

In this course, a *k-algebra* A means an associative but not necessarily commutative ring with unit, equipped with a unital ring homomorphism from k to the center $Z(A)$ of A, where k is a commutative ring with unit (often a field). By *A-module* we always mean unital left A-module. Right A-modules are then A^{op} modules, where $A^{\mathrm{op}} = A$ as a k-module and has the opposite multiplication as an algebra.

1. Given rings with unit A and k, where k is commutative, show that to give a k-alebra structure on A, *i.e.*, a homomorphism $k \to Z(A)$, it is equivalent to give a k-module structure on A such that multiplication in A is k-bilinear. (In the language of multilinear algebra, this means that the multiplication in A is given by a k-module homomorphism $A \otimes_k A \to A$.)

2. Let A be a k-algebra, and let G be a group acting on A by k-algebra automorphisms (thus, acting trivially on the image of k in A). Define the twisted group algebra, or 'smash product,' $A * G$ to be the free A-module with basis G, with multiplication defined by

$$\left(\sum_g a_g g\right)\left(\sum_h b_h h\right) = \left(\sum_{g,h} a_g g(b_h)\, gh\right).$$

(a) Verify that $A*G$ is an associative k-algebra with unit 1_G, and that its given A-module structure coincides with the one in which scalar multiplication is left multiplication in $A*G$ by the subalgebra $A \cdot 1_G \cong A$.

(b) A *G-equivariant A-module* is an A-module V with a k-linear G action, such that scalar multiplication $A \times V \to V$ is G-invariant (i.e., intertwines the G actions on $A \times V$ and V). Show that every G-equivariant A-module V has a canonical structure of $A*G$-module, and vice versa, in such a way that G-invariant A-module homomorphisms between equivariant A-modules are the same as $A * G$-module homomorphisms.

(c) In the case $A = k$ (in particular, G acting trivially on A), show that $A * G$ reduces to the usual group algebra kG, and (b) gives the identification of G-modules over k with kG-modules.

3. Let $G = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n. Let $X = \{0, \ldots, n-1\}$ and let $\mathbb{C}^X$ be the $\mathbb{C}$-algebra of $\mathbb{C}$-valued functions on X. Show that there is an isomorphism $\mathbb{C}G \cong \mathbb{C}^X$ (called *discrete Fourier transform*) sending a generator of G to the function $f(x) = e^{2\pi i x/n}$. More generally, if k is a field and $\omega \in k$ is a primitive n-th root of unity, construct a similar isomorphism $kG \cong k^X$.

4. With $G = \mathbb{Z}/n\mathbb{Z}$ again, describe the group algebra kG when k is an algebraically closed field of characteristic p dividing n (so all roots of unity have order coprime to p and hence not equal to n). Begin by showing that in the case $n = p$, we have $kG \cong k[t]/(t^p)$.

Lecture 3

1. Let $V = k^n$ be the standard representation of S_n over a field k of characteristic p dividing n, let U be the submodule generated by the vector $(1, 1, \ldots, 1)$, and let W be the submodule consisting of all vectors $\mathbf{z}$ such that $\sum_i z_i = 0$. Prove that W/U is irreducible, and deduce that

$$0 \subset U \subset W \subset V$$

is a composition series for V. Describe the composition factors.

2. With notation as in the previous problem, prove that U and W are the only proper non-zero submodules of V. In particular, V is indecomposable.

Lecture 4

1. Let $A = M_n(k) = \mathrm{End}_k(k^n)$, where k is a field (or, more generally, a division ring, if you want to work a little harder).

(a) Prove that the natural A-module $V = k^n$ is simple.

(b) Construct a composition series for A as a left A-module.

(c) Use (b) to deduce that V is the unique simple A-module, up to isomorphism.

(d) Prove that $\mathrm{End}_A(V)$ is isomorphic to k, or, in the case that k is a non-commutative division ring, to k^{op}.

2. Let k be a field and let A be the (commutative) algebra of functions k^X, where X is an infinite set. Show that the Jacobson radical $J(A)$ is zero. Prove that if $I, J \subseteq A$ are ideals such that $A = I \oplus J$, then there is a subset $Y \subseteq X$ such that I consists of all functions vanishing on Y and J consists of all functions vanishing on $X \setminus Y$. Deduce that A is not semisimple as an A-module. Hint: the functions of finite support form an ideal. Or you can use the fact that A is not Artinian.

3. Let k be a field and let A be the endomorphism algebra of an infinite-dimensional vector space V over k. Prove that V is a simple module and hence $J(A) = 0$. Prove that A is not Artinian and hence not semisimple as an A-module.

4. Let $\mathbb{H}$ denote the algebra of quaternions. Prove that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$ is semisimple, hence isomorphic to a direct product of matrix algebras over division rings over $\mathbb{R}$. Construct such an isomorphism explicitly.

Lectures 5-8

1. [Fulton-Harris, 1.3] (a) Show that if V is a G-module, the exterior powers $\bigwedge^k V$ have natural G actions such that $g(v_1 \wedge \cdots \wedge v_k) = g(v_1) \wedge \cdots \wedge g(v_k)$.

(b) Suppose that every $g \in G$ acts on V by an endomorphism $\rho(g)$ of determinant 1, *i.e.*, the action $\rho\colon G \to GL(V)$ factors through $SL(V)$. Show that $\bigwedge^k V$ is isomorphic to $\bigwedge^{n-k}(V^*)$, where $n = \dim V$.

2. Let $V = \mathbb{C}^n$ be the standard representation of S_n and $W = V/V^{S_n}$ its irreducible quotient of dimension $n-1$. Show that the exterior powers $\bigwedge^k(W)$ are irreducible and mutually non-isomorphic. What is a simple description of $\bigwedge^{n-1}(W)$?

3. [Fulton-Harris, 1.14] Let V be a finite-dimensional complex representation of a finite group G. Prove that there exists a G-invariant non-degenerate Hermitian form on V; equivalently there exists a basis of V in which G acts by unitary matrices. Prove that if V is irreducible, the invariant form is unique up to a scalar factor.

4. [Fulton-Harris, 2.35] Recall that for $k = \mathbb{C}$, the algebraic inner product of characters coincides with the Hermitian inner product

$$(\phi, \chi) = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\chi(g)}$$

on the space of functions $\mathbb{C}^G$. Suppose that we choose one irreducible G-module V_i from each isomorphism class, and in each of these we choose a basis in which G acts by unitary matrices $A_i(g)$ (see the previous problem). Prove that the rescaled matrix entries $\sqrt{\dim V_i}\, A_i(g)_{j,k}$, for all i, j, k, form an orthonormal basis of $\mathbb{C}^G$.

5. [Fulton-Harris, 2.1-2.2] Show that the characters of the second symmetric and exterior powers of V are given in terms of the character of V by

$$\chi_{\bigwedge^2 V}(g) = (\chi_V(g)^2 - \chi_V(g^2))/2$$
$$\chi_{S^2 V}(g) = (\chi_V(g)^2 + \chi_V(g^2))/2$$

6. Assume $k = \overline{k}$ and $\operatorname{char} k = 0$. The *regular representation* of G is kG, regarded as a kG-module. Show in two ways that each irreducible G-module V occurs with multiplicity $\dim(V)$ in kG: (i) using the structure of semisimple algebras; (ii) using characters.

7. Assume $k = \overline{k}$ and char $k = 0$. Prove that if V is a faithful G-module, then every simple G-module occurs in some symmetric power of V. Hint: think of the symmetric algebra $S(V)$ as the algebra of polynomial functions on V^*, and show that there is a quotient $S(V)/I$, where I is a G-invariant ideal, such that $S(V)/I$ is isomorphic to kG as a G-module.

8. [Fulton-Harris, 2.39]. Show that if V is irreducible and $\dim(V) > 1$ then $\chi_V(g) = 0$ for some $g \in G$.

9. Prove that if A and B are finite-dimensional semisimple algebras over an algebraically closed field k, and V, W are simple A and B modules, respectively, then $V \otimes_k W$ is a simple $A \otimes_k B$ module. Prove, moreover, that as V and W range over isomorphism types of simple modules, $V \otimes_k W$ gives all the distinct simple $A \otimes_k B$ modules up to isomorphism.

Deduce (for k algebraically closed and char $k = 0$) that the irreducible characters of $G \times H$ are given by $\chi_{i,j}(g, h) = \phi_i(g)\psi_j(h)$, where ϕ_i and ψ_j are the irreducible characters of G and H, respectively.

10. In this problem we take characters over the complex numbers $\mathbb{C}$.

(a) Prove that, for every character χ_V of G, the set $\{g \in G | \chi(g) = \chi(1)\}$ is the kernel of the action $G \to GL(V)$. In particular, it is a normal subgroup of G.

(b) Prove that every normal subgroup of G is an intersection of kernels of actions of G on irreducible representations.

(c) Deduce that G is simple if and only if $\chi(g) \neq \chi(1)$, for every $1 \neq g \in G$ and every irreducible character $\chi \neq 1$.

11. Goodman and Wallach Ex. 4.4 #2 (p. 223). Note: the Goodman and Wallach text is available online through the UC library.

12. Goodman and Wallach Ex. 4.4 #3 (p. 223).

Lectures 9-10

1. Calculate the character table of the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. Hint: four of the characters factor through a homomorphism $Q \to (\mathbb{Z}/2\mathbb{Z})^2$.

2. Calculate the character table of the dihedral group D_8 of order 8. See hint for the preceding problem.

3. Show that the real group algebra $\mathbb{R}Q$ has four irreducible representations of dimension 1 and one of dimension 4, the last being the 2-dimensional representation of $\mathbb{C}Q$, which remains irreducible when restricted to $\mathbb{R}Q$, and which can also be described as the action of Q by left multiplication in $\mathbb{H}$.

4. Show that the real group algebra $\mathbb{R}D_8$ has four irreducible representation of dimension 1 and one of dimension 2, that inducing these from $\mathbb{R}D_8$ to $\mathbb{C}D_8$ gives the irreducible complex representations of D_8, each of which restricts to two copies of an irreducible representation of $\mathbb{R}D_8$.

5. Let D_{2n} be the dihedral group of order $2n$, and assume $n > 2$ (since D_2 and D_4 are abelian).

(a) Show that the defining representation of D_{2n} by reflections and rotations in $\mathbb{R}^2$, when complexified, gives a 2-dimensional irreducible representation V of D_{2n} over $\mathbb{C}$.

(b) Show that if $n = 2k$, then D_{2n} has four 1-dimensional irreducible representations and $k-1$ 2-dimensional ones; while if $n = 2k+1$, then D_{2n} has two 1-dimesional irreducibles and k 2-dimensional ones. In either case, show that all the 2-dimensional representations can be obtained from V or V^* by composing the action $\rho\colon D_{2n} \to GL(V)$ or $\rho\colon D_{2n} \to GL(V^*)$ with an endomorphism of D_{2n}.

(c) Construct the character table of D_{2n} (it helps to do this simultaneously with part (b)).

Lecture 11

1. Let G be a finite group and $H \subseteq G$ a subgroup of index 2 (in particular, H is normal). Then $G/H \cong \mathbb{Z}/2\mathbb{Z}$ has a nontrivial 1-dimensional character ϵ with values ± 1, which we may regard as a character of G. Note that G/H acts on the set of conjugacy classes of H and therefore on the set of class functions on H.

(a) Prove that the action of G/H on class functions on H sends irreducible characters to irreducible characters. (More generally, for any normal subgroup H of a finite group G, G acts on H by automorphisms, and so acts on the characters of H. Since H fixes its own characters, G/H acts.)

(b) Prove that each conjugacy class C of G which is contained in H is either a conjugacy class of H or a union of two conjugacy classes of H, with the latter occurring if and only if the centralizer in G of any element of C is contained in H. Also verify that the action of G/H on the set of conjugacy classes of H fixes those which are classes of G, and switches the two classes of H in each class of G which is not a class of H.

(b) Let χ be an irreducible character of G such that $\chi \otimes \epsilon = \chi$. Prove that $\operatorname{Res}^G_H \chi$ is a sum of two irreducible characters of H which are exchanged by the action of G/H.

(c) Let χ be an irreducible character of G such that $\chi' = \chi \otimes \epsilon \neq \chi$. Prove that $\operatorname{Res}^G_H \chi = \operatorname{Res}^G_H \chi'$, and that this is an irreducible character of H.

(d) Let χ be an irreducible character of H fixed by G/H. Prove that $\operatorname{Ind}^G_H \chi$ is a sum of two distinct irreducible characters $\phi, \phi' = \phi \otimes \epsilon$ of G.

(e) Let χ, χ' be two irreducible characters of H exchanged by G/H. Prove that $\operatorname{Ind}^G_H \chi = \operatorname{Ind}^G_H \chi'$, and that this is an irreducible character of G.

(f) Prove that parts (b) through (e) set up a 1-1 corresponsdence between G/H orbits on the set of irreducible characters of H, and orbits of tensoring with ϵ on the set of irreducible characters of G (note that $\epsilon \otimes \epsilon = 1$), such that orbits with 2 elements on either side correspond to fixed points on the other side.

2. Use the preceding problem to classify the conjugacy classes and irreducible characters of the alternating groups A_n, and express the character values in terms of characters of the symmetric group S_n. Specifically, work out the character table of A_5.

3. Let $p(n)$ denote the number of partitions of n, $k(n)$ the number of self-conjugate partitions, and $e(n)$ the number of partitions with an even number of even parts.

(a) Show that the number of partitions of n with distinct odd parts is equal to $k(n)$.

(b) Prove that $e(n) + k(n) = (p(n) + 3k(n))/2$. Explain why I wrote the identity this way instead of as $e(n) = (p(n) + k(n))/2$.

Lectures 12-18

1. Identify the partitions λ for which the irreducible representation V_λ in the standard classification of irreducible representations of S_n is isomorphic to the k-th exterior power of the $n-1$ dimensional irreducible submodule of the defining representation $\mathbb{C}^n$.

2. Let S_n act on the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$ by permuting the variables. Let R_d denote the subspace consisting of homogeneous polynomials of degree d.

(a) Use the fact that S_n permutes monomials to express the Frobenius image $F\chi(R_d)$ of the character of R_d in terms of complete homogeneous symmetric functions.

(b) Prove that the generating function for these characters is given by the formula

$$\sum_d F\chi(R_d)\, q^d = \sum_{|\lambda|=n} s_\lambda(1/(1-q)) s_\lambda(x),$$

where $s_\lambda(1/(1-q))$ is shorthand for $s_\lambda(1, q, q^2, \ldots)$, a formal power series in q (which happens to be rational function of q; see the next problem).

3. If $f(x)$ is a symmetric function, let $f(x/(1-q))$ denote the symmetric function with coefficients in $\mathbb{Q}(q)$ which results by writing f in terms of power-sums and substituting $p_k/(1-q^k)$ for p_k, for all k.

(a) Show that f evaluated on the alphabet $\{x_i q^j\}$ for all i and all $j \geq 0$, regarded as a symmetric function in the x_i with coefficients in the ring of formal power series in q, has the property that its coefficients are actually rational functions of q, and as such it coincides with $f(x/(1-q))$.

(b) Show that the formula in the previous problem for the generating for the characters of R_d can also be written as $h_n(x/(1-q))$.

(c) More generally, show that if V is an S_n module, and $f(x) = F\chi(V)$ is the Frobenius image of its character, then the Frobenius image of the character of the $V \otimes R_d$ is given by

$$\sum_d F\chi(V \otimes R_d)\, q^d = f(x/(1-q)).$$

4. If $f(x)$ is a symmetric function, let $f(x(1-q))$ denote the symmetric function with coefficients in $\mathbb{Q}(q)$ which results by writing f in terms of power-sums and substituting $(1-q^k)p_k$ for p_k, for all k.

(a) Show that if V is an S_n module, and $f(x) = F\chi(V)$ is the Frobenius image of its character, then the Frobenius image of the character of the $V \otimes \bigwedge^d \mathbb{C}^n$ is given by

$$\sum_d F\chi(V \otimes \bigwedge^d \mathbb{C}^n)\, q^d = f(x(1-q)).$$

5. Calculate the matrices of the transpositions $s_i = (i,\, i{+}1)$ in the irreducible representation $V_{(3,2)}$ of S_5, with respect to the basis corresponding to standard Young tableaux. Check your answer by verifying that your matrices satisfy the relations $s_i s_j = s_j s_i$ for $|i-j|>1$, $s_i s_j s_i = s_j s_i s_j$ for $j = i+1$.

6. (a) Let p_k denote a power-sum symmetric function and s_λ a Schur function. Prove that $p_k s_\lambda$ is a sum of Schur functions s_μ, each with coefficient ± 1, where μ ranges over partitions of $|\lambda| + k$ such that the diagram of μ contains that of λ, and the skew diagram μ/λ is a connected ribbon of size k: "ribbon" means it contains no 2×2 square. Show that the coefficient is $(-1)^{h-1}$ where h is the number of rows occupied by the ribbon.

Hint: calculate $p_k a_{\lambda+\delta}$ in terms of antisymmetric functions.

(b) Define a ribbon tableau R of shape λ to be a filling of the diagram of λ with positive integers, weakly increasing on each row and column, such that the set of boxes occupied by i is a ribbon for each i. Define the sign $\epsilon(R)$ to be the product of $(-1)^{h-1}$ over these ribbons. Deduce from part (a) that the character χ_λ of S_n, evaluated on a permutation of cycle type τ, is the sum of $\epsilon(R)$ over ribbon tableaux R of shape λ and weight τ. This combinatorial formula for the characters of S_n is known as the Murgnahan-Nakayama rule.

Lectures 19-24

1. Recall that $\mathcal{O}(GL_n)$ is generated by the matrix entries of $X \in GL_n$ together with $1/\det(X)$. An algebraic representation $\rho\colon GL_n \to GL_m$ is called *polynomial* if the matrix entries of $\rho(X)$ are polynomials in the matrix entries of X, *i.e.*, they can be written without using $1/\det(X)$.

(a) Show that every submodule of a tensor power of the defining representation $\mathbb{C}^n$ of GL_n is a polynomial representation.

(b) Show that every algebraic representation of GL_n is the tensor product of a polynomial representation and some power of the 1-dimensional represenation $X \mapsto 1/\det(X)$.

(c) [for Lecture 38] Prove the converse to (a), *i.e.*, every irreducible polynomial representation of GL_n occurs in a tensor power of the defining representation.

2. For any integer m we can make the group $G_m = \mathbb{C}^\times$ act on group $G_a = (\mathbb{C}, +)$, with t acting as scalar multiplication by t^m. Let G be their semidirect product, identified as an algebraic variety with the product, *i.e.*, $\mathcal{O}(G) = \mathbb{C}[x, t^{\pm 1}]$, where x is the coordinate on G_a and t is the coordinate on G_m.

(a) Show that G is an algebraic group, and write down the comultipication and antipode (corresponding to the inverse map) on $\mathcal{O}(G)$.

(b) Show that for $m = 2$, G is isomorphic to the subgroup of upper triangular matrices in SL_2.

(c) Show that an algebraic representation of G is the same thing as a finite-dimensional vector space V equipped with a $\mathbb{Z}$-grading and an endomorphism $\alpha \in \mathrm{End}_{\mathbb{C}}(V)$ homogeneous of degree m. More generally and precisely, show that the category of of $\mathcal{O}(G)$ comodules is equivalent to the category of $\mathbb{Z}$-graded vector spaces with a locally nilpotent degree m endomorphism.

3. What are the unipotent radical and reductive quotient of the group G in the preceding problem?

4. Let positive integers r_i be given such that $r_1 + \cdots + r_k = n$. Let $P \subseteq GL_n$ be the subgroup consisting of matrices which are block upper-triangular with block sizes $r_1, \ldots, r_k$. Describe the unipotent radical and reductive quotient of the algebraic group P.

5. Let $\rho\colon G \times X \to X$ be an algebraic action of an algebraic group G on a variety X. For simplicity, assume X is affine. Let $\rho^\sharp\colon \mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{O}(G)$ be the ring homomorphism corresponding to the action. Given $\xi \in \mathrm{Lie}(G) \subseteq \mathcal{O}(G)^*$, we get a linear map $L_\xi = (1 \otimes \xi) \circ \rho^\sharp\colon \mathcal{O}(X) \to \mathcal{O}(X)$.

Prove that: (i) L_ξ is a derivation of $\mathcal{O}(X)$, *i.e.*, a vector field on X; (ii) denoting by $L_\xi(x)$ the tangent vector at $x \in X$ given by the vector field to L_ξ at x, the map $\xi \mapsto L_\xi(x)$ is the differential at 1 of the orbit map $G \to X$, $g \mapsto gx$; and (iii) $\xi \mapsto -L_\xi$ is a Lie algebra homomorphism, where we regard $\mathrm{Der}(\mathcal{O}(X))$ as a Lie algebra under commutator of derivations.

Hint: the minus sign in (iii) comes from the conventional identification of the Lie bracket in $\mathrm{Lie}(G)$ with the commutator of left invariant vector fields. If we instead identify $\mathrm{Lie}(G)$ with the set of right invariant vector fields, it reverses the Lie bracket. For the left action of G on itself, L_ξ is the right invariant vector field on G such that $L_\xi(1) = \xi$.