Math 249—Spring 2024
Homework problems on Lectures 16–31

1. Express $m_\lambda(1, 1, \ldots, 1)$, with n ones, as a more familiar combinatorial quantity.

2. Prove the formula for complete homogeneous symmetric functions in terms of elementary symmetric functions

$$h_n = \sum_{|\lambda|=n} (-1)^{n-l(\lambda)} \binom{l(\lambda)}{r_1, r_2, \ldots, r_k} e_\lambda,$$

where $\lambda = (1^{r_1}, 2^{r_2}, \ldots, k^{r_k})$.

3. Prove the formulas for the n-th power-sum symmetric function

$$p_n = \sum_{k=0}^{n-1} (-1)^k s_{(n-k,1^k)} = \sum_{k=0}^{n-1} (-1)^k (n-k) h_{n-k} e_k = \sum_{k=1}^{n} (-1)^{k-1} k h_{n-k} e_k.$$

4. In class we defined the skew Schur function

$$s_\nu(X) = \sum_{T \in \mathrm{SSYT}(\nu)} x^T$$

to be a generating function for semistandard Young tableaux on a skew Young diagram ν. Show that for $\nu = \lambda/\mu$ expressed as a difference of partition diagrams, s_ν is characterized by the property

$$s_{\lambda/\mu} = s_\mu^\perp s_\lambda, \quad \text{that is,} \quad \langle s_{\lambda/\mu}, f \rangle = \langle s_\lambda, s_\mu f \rangle \quad \text{for all } f,$$

where $\langle -, - \rangle$ is the Hall inner product.

5. Let $\lambda = (k, 1^l)$ be a hook shape. Prove the following rule for computing $s_\lambda s_\mu$, which generalizes the Pieri rules for $s_{(k)} s_\mu$ and $s_{(1^k)} s_\mu$:

(i) s_ν occurs with non-zero coefficient in $s_\lambda s_\mu$ only if ν/μ is a disjoint union of ribbons (connected skew shapes containing no 2×2 rectangle) of total size $k + l$,

(ii) in that case, the coefficient is $\binom{r-1}{h-l-1}$, where μ/ν consists of r ribbons, and h is the sum of their heights (interpreting $\binom{r-1}{h-l-1}$ as zero if $h - l - 1$ is negative).

6. Prove that the Frobenius characteristic map is given in terms of monomial symmetric functions by

$$F(\chi_V) = \sum_\mu \dim(V^{S_\mu})\, m_\mu,$$

where χ_V is the character of an S_n module V, S_μ is the Young subgroup $S_{\mu_1} \times \cdots \times S_{\mu_l} \subseteq S_n$, and $V^{S_\mu} \subseteq V$ denotes the subspace of elements invariant under the action of S_μ.

7. Let V_λ be the irreducible representation of S_n whose character corresponds under the Frobenius characteristic map to the Schur function s_λ.

(a) Prove that $V_{(n-1,1)}$ is the irreducible submodule of dimension of dimension $n-1$ in the defining representation of S_n on $\mathbb{C}^n$, complementary to the one copy of the trivial representation in $\mathbb{C}^n$.

(b) Prove that $V_{(n-k,1^k)}$ is isomorphic to the k-th exterior power of $V_{(n-1,1)}$.

8. Let T be a standard tableau of straight shape, i.e., the shape is a partition diagram. Schützenzerger's *evacuation* $\mathrm{ev}(T)$ is defined as follows: rotate T 180° and renumber the entries $1,\ldots,n$ to $n,\ldots,1$, to obtain a standard tableau of 'anti-straight' shape, then rectify.

Let $w \mapsto (P(w), Q(w))$ be the RSK correspondence restricted to permutations w, so both $P(w)$ and $Q(w)$ are standard tableaux.

(a) Show that the reversed permutation $w'(i) = w(n+1-i)$ goes under RSK to $P(w') = P(w)^*$, $Q(w') = \mathrm{ev}(Q(w)^*) = (\mathrm{ev}\, Q(w))^*$, where S^* denotes the transpose of a standard tableau.

(b) Show that the permutation with reversed values $w''(i) = n+1-w(i)$ goes to $P(w'') = \mathrm{ev}(P(w)^*)$, $Q(w'') = Q(w)^*$.

(c) Show that the inverse permutation w^{-1} goes to $P(w^{-1}) = Q(w)$, $Q(w^{-1}) = P(w)$. [N.B.: this part is more challenging, and implies that parts (a) and (b) are equivalent.]