Math 249—Spring 2024
Homework problems on Lectures 7–15

1. The *Hadamard product* $F * G$ of two species is the species $(F * G)(S) = F(S) \times G(S)$. Show that the species $L * L$ and $P * L$ are equivalent. This can be understood as an explanation of the fact that the inequivalent species L and P have the same exponential generating function.

2. Given a species F, define $F'(S)$ to be the set of F structures on the set $S \coprod \{*\}$ given by adjoining a new distinguished element to S. For example, if F is the species of unrooted trees, then F' is the species of rooted forests.

(a) Show that the exponential generating function for F' is the derivative $F(x)'$ of the exponential generating function for F.

(b) Describe F' when F is the species L of linear orderings, find both exponential generating functions from other principles, and check part (a) in this case.

3. A *distribution* is a function together with a linear ordering on the preimage of each element of the codomain. Let J_k be the species such that $J_k(S)$ is the set of distributions from S to $\{1, \ldots, k\}$.

(a) Find the exponential generating function for J_k, and from it obtain the formula $(n + k - 1)_n$ for the number of distributions from an n element set to a k element set.

(b) Do the same for the species of *surjective* distributions, obtaining the formula

$$(n)_k(n-1)_{n-k}$$

for the number of them.

4. An *unordered binary tree* is a rooted tree in which each non-leaf node has two children, but we do not order the children. Find the exponential generating function which counts unordered binary trees on n labelled nodes.

5. A *perfect matching* on a set S of $2n$ elements is a partition of S into n blocks of two elements each. Perfect matchings form a species M, with $M(S) = \emptyset$ if $|S|$ is odd.

(a) Find the exponential generating function for the species of perfect matchings.

(b) Deduce algebraically that the number of perfect matchings on a set of $2n$ elements is $n!!$. Here and below the 'double factorial' notation $n!!$ stands for the product $(2n-1)(2n-3)\cdots 3 \cdot 1$ of the first n odd numbers.

(c) Give a direct counting argument for the result in (b).

6. Let $e(2n)$ be the number of permutations σ of a set of $2n$ elements with the property that every cycle of σ has even length.

(a) Find the exponential generating function $\sum_n e(2n)x^{2n}/(2n)!$.

(b) Deduce that $e(2n) = (n!!)^2$.

7. (a) From the preceding problems it follows that the number of permutations of a $2n$ element set S with only even cycles is equal to the number of pairs of perfect matchings on

S. Construct a direct bijection between the two (to do this you will probably need to fix the set S to be $[2n]$ and use the numerical values of its elements to make some auxiliary choices).

(b) Show that the species of permutations with even-length cycles and the species of pairs of perfect matchings are not equivalent. This can be understood as explaining the need for auxiliary choices in part (a).

8. (a) Find the exponential generating function $\sum_n r_n x^n/n!$, where r_n is the number of permutations of odd order of an n element set.

(b) Deduce that $r_{2n} = (n!!)^2$ and $r_{2n+1} = (2n+1)(n!!)^2$.

9. Let $g(n,k)$ denote the number of connected simple graphs (i.e., without loops or multiple edges) on n labelled vertices with k edges. Derive the mixed ordinary/exponential generating function

$$\sum_{n=1}^{\infty} \sum_k g(n,k) q^k x^n/n! = \log \sum_{n=0}^{\infty} (1+q)^{\binom{n}{2}} x^n/n!$$

and use it to compute $\sum_k g(n,k) q^k$ for all $n \leq 4$. As a check, count the graphs in question by hand and compare answers.

10. Let $g_+(n)$, $g_-(n)$ denote the number of connected simple graphs on the vertex set $[n]$ with an even or an odd number of edges, respectively. Prove that $g_+(n) - g_-(n) = (-1)^{n-1}(n-1)!$.

11. (a) The *diameter* d of a tree T is the maximum length of a path in T (a path of length n has n edges and $n+1$ vertices). Prove that if d is even then all paths of length d have the same middle vertex, called the *center* of T, and if d is odd, then all paths of length d have the same middle edge, called the *bicenter* of T.

(b) Show that if d is even, the species of labelled unrooted trees of diameter d is equivalent to the species of labelled rooted trees of height $d/2$ with the property that at least two children of the root are roots of subtrees of height $d/2 - 1$.

(c) Show that if d is odd, the species of labelled unrooted trees of diameter d is equivalent to the species of unordered pairs of disjoint rooted trees of height $(d-1)/2$.

(d) Let T_h be the species of labelled rooted trees of height h, and let $T_{\leq h} = T_0 + \cdots + T_h$. Show that these are given by the recurrence

$$\begin{aligned} T_0 &= X \\ T_h &= X((E-1) \circ T_{h-1})(E \circ T_{\leq h-2}) \quad \text{for } h > 0. \end{aligned}$$

(e) Use (a), (b) and (c) to express the species U_d of labelled unrooted trees of diameter d in terms the species T_h.

(f) Use (d) and (e) to calculate the number of labelled unrooted trees of diameter d on n vertices, for $n \leq 5$ and all d. Check that your answers summed over d agree with the known formula n^{n-2}.

(g) Use (d) and (e) to calculate the number of unlabelled unrooted trees of diameter d on n vertices, for $n \le 5$ and all d. Check your answers by listing the trees.

12. (a) Find an explicit formula for the cycle index Z_I of the species of involutions, $I(S) = \{\sigma \in \mathfrak{S}_S : \sigma^2 = 1\}$.

(b) Evaluate $Z_I[x]$ and verify that it agrees with the obvious ordinary generating function counting involutions up to conjugacy.

13. In the lecture we proved the following version of Cayley's formula:

$$\sum_T \prod_j x_j^{c_T(j)} = (x_1 + \cdots + x_n)^{n-1},$$

where T ranges over rooted trees on $[n]$, and $c_T(j)$ is the number of children of vertex j in T.

(a) Prove the following variant of Cayley's formula:

$$\sum_T \prod_j x_j^{d_T(j)} = x_1 \cdots x_n (x_1 + \cdots + x_n)^{n-2},$$

where T ranges over *unrooted* trees on $[n]$, and $d_T(j)$ is the degree (i.e., number of neighboring vertices) of vertex j in T.

(b) Show that (a) implies that the sum $\sum_T \prod_j x_j^{c_T(j)}$ over trees with root i is equal to $x_i(x_1 + \cdots + x_n)^{n-2}$, which implies the version of Cayley's formula we proved in the lecture.

14. (a) From Cayley's tree generating function derive the identity

$$\sum_F \prod_{i=1}^{n} x_i^{c_F(i)} = \binom{n-1}{k-1} (x_1 + \cdots + x_n)^{n-k},$$

where the sum is over rooted forests F with k components on vertices $\{1, \ldots, n\}$, and $c_F(i)$ denotes the number of children of vertex i in F.

(b) Deduce the identity

$$\sum_F \prod_{i=1}^{n} h_{c_F(i)} = \binom{n-1}{k-1} \langle \frac{x^{n-k}}{(n-k)!} \rangle H(x)^n,$$

where $H(x) = \sum_m h_m x^m/m!$ is a generic formal power series written in exponential form and $\langle \cdot \rangle$ denotes taking a coefficient.

(c) Let H be the 'generic species' with exponential generating function $H(x)$, that is, the trivial species, but enumerated by assigning weight h_n to the one structure on any set with n elements. Let $F(x)$ be the solution of the formal functional equation

$$F(x) = x\, H(F(x))$$

(that is, assuming h_0 invertible, the functional composition inverse of $x/H(x)$). Using (b) and a species interpretation of $F(x)$, obtain the generalized Lagrange inversion formula

$$\langle \frac{x^n}{n!} \rangle F(x)^k/k! = \binom{n-1}{k-1} \langle \frac{x^{n-k}}{(n-k)!} \rangle H(x)^n,$$

or equivalently,

$$\langle x^n \rangle F(x)^k = \frac{k}{n} \langle x^{n-k} \rangle H(x)^n.$$