Math 249—Spring 2024
Homework problems on Lectures 1–6

1. Defining the derivative $\frac{d}{dx}F(x)$ of a formal power series term by term in the obvious way, show that the usual sum and product rules and the chain rule hold. Show that Taylor's formula holds if the coefficient ring contains $\mathbb{Q}$.

2. Assume we are working in a formal power series ring $R[[x]]$ over a coefficient ring R containing $\mathbb{Q}$. Define $\exp(x)$ and $\log(1+x)$ by their usual Taylor series.

(a) Given $F(x), G(x) \in R[[x]]$ such that $F(0) = 1$, show that there is a well-defined formal power series

$$F(x)^{G(x)} \underset{\text{def}}{=} \exp(G(x) \log F(x)).$$

(b) Show that the above definition satisfies the laws of exponents $F(x)^{G(x)+H(x)} = F(x)^{G(x)}F(x)^{H(x)}$, $F(x)^{G(x)H(x)} = (F(x)^{G(x)})^{H(x)}$, $F(x)^0 = 1$, $F(x)^1 = F(x)$.

3. Generalizing the definition in the previous problem in the obvious way to formal series in more than one variable, show that (over a coefficient ring containing $\mathbb{Q}$) the Newton binomial theorem

$$(1+x)^y = \sum_n \binom{y}{n} x^n$$

holds as a formal power series identity.

4. Express each of the following as a binomial coefficient: (a) the number of monomials of degree exactly d in a (commutative) polynomial ring in n variables; (b) the number of monomials of degree $\leq d$.

5. Show that $\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} \left\langle {n-1 \atop j} \right\rangle$. Using this, evaluate $\sum_{j=0}^{k} \binom{n+j}{j}$ as a single binomial coefficient.

6. (a) Give two proofs of the binomial coefficient identity, called the *convolution formula*,

$$\sum_j \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k}.$$

One proof should use generating functions, the other should be a direct combinatorial proof.

(b) Discover and prove in the same two ways an analogous identity for multiset coefficients $\left\langle {n \atop k} \right\rangle$.

7. Find a closed expression for the ordinary generating function in two variables

$$\sum_{n,k \geq 0} \binom{n}{k} x^n y^k,$$

and use it to deduce the identity

$$\sum_r \binom{r}{k} x^r = \frac{x^k}{(1-x)^{k+1}}.$$

8. Find a closed expression for the two-variable ordinary generating function

$$\sum_{n,k\geq 0} \left\langle {n \atop k} \right\rangle x^n y^k,$$

9. Find closed expressions for the mixed ordinary/exponential generating functions

$$\sum_{k,n} S(n,k) y^k \frac{x^n}{n!}\,, \qquad \sum_{k,n} c(n,k) y^k \frac{x^n}{n!}$$

for Stirling numbers $S(n,k)$ and signless Stirling numbers $c(n,k)$.

10. By expanding the right hand side in partial fractions, show directly that the generating function for Stirling numbers

$$\sum_n S(n,k) x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-k\,x)}$$

is equivalent to the explicit formula

$$S(n,k) = \frac{1}{k!} \sum_j (-1)^{k-j} \binom{k}{j} j^n.$$

11. Let $m_d(q)$ be the number of irreducible monic polynomials $f(x)$ of degree d, over the finite field $\mathbb{F}(q)$ with q elements. Note that the number of *all* monic polynomials of degree d (irreducible or not) is just q^d.

(a) Use unique factorization of polynomials to prove the generating function identity

$$\prod_{d\geq 1} \frac{1}{(1-x^d)^{m_d(q)}} = \frac{1}{1-qx}.$$

(b) By taking logarithms on both sides, derive the identity

$$\sum_{d|n} d m_d(q) = q^n$$

for all n, the sum ranging over the divisors of n. Equivalently,

$$m_d(q) = \frac{1}{d} \sum_{m|d} \mu(d/m) q^m,$$

where $\mu(n)$ is the Möbius function from number theory, i.e., $\mu(n) = (-1)^k$ if n is a product of k distinct primes, and $\mu(n) = 0$ if n is divisible by a square.

(c) Use (b) to prove that the product of all monic irreducible polynomials of degree dividing n is equal to $x^{q^n} - x$. [Hint: every element of $\mathbb{F}(q^n)$ is a root of $x^{q^n} - x$.]

(d) A *necklace* is an equivalence class of words up to rotation. A necklace of length n is *primitive* if the corresponding rotation class consists of n distinct words, i.e., it is not periodic with period d a proper divisor of n. (Example: 1122 is primitive; 1212 is not.) Note that every word of length n consists of n/d repetitions of a primitive necklace of length d dividing n. (Example: 1212 and 2121 both repeat the primitive necklace $12 = 21$.) Let $p_d(q)$ be the number of primitive necklaces of length d on an alphabet of q symbols. Prove that

$$\sum_{d|n} d p_d(q) = q^n,$$

and hence

$$m_d(q) = p_d(q)$$

when q is a power of a prime.

(e) (For those familiar with Lie algebras.) Let $L_q = L(x_1, \ldots, x_q)$ be the free Lie algebra with q generators. L_q is graded if we consider each generator to be homogeneous of degree 1. Let $l_d(q)$ be the dimension of the homogeneous part of degree d in L_q. In other words, $l_d(q)$ is the number of linearly independent expressions that can be formed by bracketing together d of the generators x_i. The universal enveloping algebra of L_q is the free tensor algebra $T(x_1, \ldots, x_q)$. Using this and the Poincaré-Birkhoff-Witt theorem, derive the identity

$$\prod_{d \geq 1} \frac{1}{(1 - x^d)^{l_d(q)}} = \frac{1}{1 - qx}.$$

Deduce that $l_d(q) = p_d(q)$.

Remark: This enumerative result suggests that L_q should have a basis whose elements in degree d are indexed in some natural way by primitive necklaces of length d. Such a basis has been constructed by R. Lyndon.

12. Prove that the q-multinomial coefficients satisfy the following recurrence. A combinatorial proof is preferred.

$$\binom{n}{k_1, k_2, \ldots, k_r}_q = \binom{n-1}{k_1 - 1, k_2, \ldots, k_r}_q + q^{k_1} \binom{n-1}{k_1, k_2 - 1, \ldots, k_r}_q + \cdots + q^{k_1 + \cdots + k_{r-1}} \binom{n-1}{k_1, k_2, \ldots, k_r - 1}_q.$$

13. Prove the following q-analog of the convolution formula for binomial coefficients. A combinatorial proof is preferred.

$$\binom{m+n}{k}_q = \sum_{i+j=k} q^{(m-i)j} \binom{m}{i}_q \binom{n}{j}_q$$

14. Let $\mathbb{Q}(q)\langle x, y\rangle$ be the algebra of polynomials in non-commuting variables x, y, over the field of rational functions $\mathbb{Q}(q)$, and let $Q_q[x, y] = \mathbb{Q}(q)\langle x, y\rangle / J$, where J is the two-sided

ideal generated by $yx-qxy$. Thus $Q_q[x,y]$ is the 'quantum polynomial ring' whose generators satisfy the q-commutation relation $yx = qxy$. Prove the 'quantum q-binomial theorem' that

$$(x+y)^n = \sum_k \binom{n}{k}_q x^k y^{n-k}$$

holds as an identity in $Q_q[x,y]$.

15. Prove that the number of partitions of n with no parts divisible by d is equal to the number of partitions of n with no part repeated d or more times, for all n and d.

16. Let $p_+(n)$ be the the number of partitions of n with an even number of parts and $p_-(n)$ the number with an odd number of parts. Let $p_{DO}(n)$ be the number of partitions of n with distinct odd parts, and let $k(n)$ be the number of partitions λ of n such that $\lambda = \lambda^*$. Prove that

$$k(n) = p_{DO}(n) = (-1)^n(p_+(n) - p_-(n)).$$

Hint: if $\lambda = \lambda^*$, dissect the diagram of λ into 'hooks' whose sizes are odd and distinct.

17. Let $k(n)$ be the number of self-conjugate partitions of n. Prove that the number of partitions of n with an even number of even parts is equal to $(p(n)+k(n))/2$. [Stanley (2nd ed.) 1.22]

18. From the q-binomial theorem

$$\prod_{i=0}^{m-1}(1+xq^i) = \sum_{j=0}^{m}\binom{m}{j}_q q^{\binom{j}{2}} x^j,$$

deduce

$$\prod_{i=1}^{s}(1+x^{-1}q^i)\prod_{i=0}^{t-1}(1+xq^i) = \sum_{j=-s}^{t}\binom{s+t}{s+j}_q q^{\binom{j}{2}} x^j.$$

By letting s and t go to infinity, prove *Jacobi's triple product identity:*

$$\sum_{j\in\mathbb{Z}}(-1)^j a^{\binom{j}{2}} x^j = \prod_{i\geq 0}(1-xa^i)(1-x^{-1}a^{i+1})(1-a^{i+1})$$

19. The *Durfee square* of a partition λ is the largest $k\times k$ square that fits inside its Young diagram. Use Durfee squares to prove the following identities [Stanley (2nd ed.), Proposition 1.8.6(b) and Ex. 1.76]:

(a)

$$\prod_{i\geq 1}\frac{1}{1-tx^i} = \sum_{k\geq 0}\frac{t^k x^{k^2}}{(1-x)\cdots(1-x^k)\,(1-tx)\cdots(1-tx^k)}$$

(b)

$$\prod_{i\geq 1}(1+tx^{2i-1}) = \sum_{k\geq 0}\frac{t^k x^{k^2}}{(1-x^2)\cdots(1-x^{2k})}$$

20. Use Durfee squares to obtain an identity analogous to part (a) of the previous problem, but enumerating partitions with distinct parts. Then deduce Sylvester's identity

$$\prod_{i\geq 1}(1-xq^i) = 1 + \sum_{n\geq 1}(-1)^n x^n \left(q^{(3n^2+n)/2} \prod_{i=1}^{n} \frac{1-xq^i}{1-q^i} + q^{(3n^2-n)/2} \prod_{i=1}^{n-1} \frac{1-xq^i}{1-q^i} \right),$$

which generalizes Euler's pentagonal number theorem.

21. Prove *Cauchy's identity:*

$$\prod_{i\geq 0} \frac{1-axq^i}{1-xq^i} = \sum_{n\geq 0} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} x^n,$$

by showing that it reduces to the q-binomial theorem upon setting $a = q^m$ for an integer m (note that you get either form of the q-binomial theorem by taking m positive or negative).

22. (a) Give a direct combinatorial proof of the partition identities

$$\prod_{i\geq 1} \frac{1}{1-xq^i} = \sum_{n\geq 0} \frac{x^n q^n}{(1-q)(1-q^2)\cdots(1-q^n)}$$

and

$$\prod_{i\geq 1}(1+xq^i) = \sum_{n\geq 0} \frac{x^n q^{\binom{n+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

(b) Show that these two identities are special cases of Cauchy's identity, above, and that they are limiting cases of the q-binomial theorem.

23. Show that the Stirling numbers of the second kind $S(n,k)$ have a q-analog $S_q(n,k)$ characterized by any of the following properties, where $[k]_q = (1-q^k)/(1-q) = 1+q+\cdots+q^{k-1}$ is the q analog of k.

(a) They satisfy the recurrence

$$S_q(n,k) = [k]_q S_q(n-1,k) + q^{k-1} S_q(n-1,k-1),$$

with inital condition $S_q(0,n) = S_q(n,0) = \delta_{0,n}$.

(b) They satisfy the following q-analog of the classical formula $x^n = \sum_k S(n,k)(x)_k$:

$$[r]_q^n = \sum_k S_q(n,k)[r]_q[r-1]_q\cdots[r-k+1]_q$$

(c) For each k, they are given by the ordinary generating function

$$\sum_n S_q(n,k)x^n = \frac{q^{\binom{k}{2}} x^k}{(1-x)(1-[2]_q x)\cdots(1-[k]_q x)}.$$

(d) Given a partition $\pi = \{B_1,\ldots,B_k\}$ of $[n]$, with the blocks numbered so that $\min(B_i) < \min(B_j)$ for $i < j$, define $\nu(\pi) = \sum_i (i-1)|B_i|$. Then $S_q(n,k) = \sum_\pi q^{\nu(\pi)}$, where the sum is over partitions of $[n]$ into k blocks.