Math 249—Spring 2020
Problem Set 2

1. We know that if $\mu \not\leq \lambda$, then $K_{\lambda\mu} = 0$, where $K_{\lambda\mu} = \langle s_\lambda, h_\mu\rangle = |SSYT(\lambda,\mu)|$ is the number of semi-standard tableaux of shape λ and weight μ. Prove that, conversely, if $\mu \leq \lambda$, then $K_{\lambda\mu} \neq 0$.

2. Prove the formula for complete homogeneous symmetric functions in terms of elementary symmetric functions

$$h_n = \sum_{|\lambda|=n} (-1)^{n-l(\lambda)} \binom{l(\lambda)}{r_1, r_2, \ldots, r_k} e_\lambda,$$

where $\lambda = (1^{r_1}, 2^{r_2}, \ldots, k^{r_k})$.

3. (a) Show that the coefficient of $m_\lambda[X]m_\nu[Y]$ in $m_\nu[X+Y]$ is equal to 1 if $\nu = \lambda \cup \mu$, and zero otherwise.

(b) Use part (a) and the fact that $\langle m_\lambda, h_\mu\rangle = \delta_{\lambda\mu}$ to show that the plethystic substitution $X \to X+Y$ is adjoint to multiplication, in the sense that for all symmetric polynomials f, g, h we have

$$\langle f, gh\rangle = \langle f[X+Y], g[X]h[Y]\rangle_{XY},$$

where $\langle -, -\rangle_{XY}$ is the inner product on $\Lambda(X) \otimes \Lambda(Y)$ defined so that $\{m_\lambda[X]m_\nu[Y]\}$ and $\{h_\lambda[X]h_\nu[Y]\}$ are dual bases.

4. Let ϵ be a fictitious alphabet such that $p_1[\epsilon] = 1$ and $p_k[\epsilon] = 0$ for $k > 1$. More properly, this means that $f[\epsilon]$ is defined to be the image of f under the homomorphism $\Lambda \to \mathbb{Q}$ mapping p_k to 1 if $k = 1$, or to 0 otherwise.

(a) Prove the identity $f[\epsilon] = \langle f, \exp(p_1)\rangle$.

(b) Prove the identity $f[\epsilon] = \lim_{n\to\infty}(f[nx])_{x\mapsto 1/n}$.

(c) Show that $e_k[\epsilon] = h_k[\epsilon] = 1/n!$.

(d) More generally, show that $s_\lambda[\epsilon] = f_\lambda/n!$, where $|\lambda| = n$ and f_λ is the number of standard Young tableaux of shape λ.

5. [From I. G. Macdonald, Symmetric Functions and Hall Polynomials]

(a) Recall from class that $h_n = \sum_{|\lambda|=n} p_\lambda/z_\lambda$, where $z_\lambda = \prod_i i^{r_i} r_i!$ for $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$. Show that this is equivalent to Newton's determinant formula

$$h_n = \frac{1}{n!}\det\begin{bmatrix} p_1 & -1 & 0 & \ldots & 0 \\ p_2 & p_1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n-1} & p_{n-2} & \cdot & \ldots & -(n-1) \\ p_n & p_{n-1} & \cdot & \ldots & p_1 \end{bmatrix}$$

(b) Show that e_n is given by the same determinant without the minus signs.

6. Let $\lambda = (l^k)$ and $\mu = (n^m)$ be partitions whose diagrams are rectangular. Suppose that $k \leq m$ and $l \leq n$, that is, the diagram of λ is contained in that of μ. Prove that $s_\lambda s_\mu = \sum_\nu s_\nu$, where ν ranges over partitions whose diagram contains the $m \times n$ rectangular diagram of μ, and the portion of the diagram of ν outside this rectangle consists of a diagram α on top of the rectangle and a diagram β to the right of the rectangle, such that α and the $180°$ rotation of β fit together in a $k \times l$ rectangle.

7. [From I. G. Macdonald, Symmetric Functions and Hall Polynomials]

Let $|\lambda| = |\mu| = n$. Show that $\langle h_\lambda, h_\mu \rangle$ is equal to the number of double cosets $S_\lambda w S_\mu$ in the symmetric group S_n, where S_λ and S_μ are Young subgroups of S_n.

8. Call a skew shape *anti-straight* if it is a $180°$ rotation of a straight (non-skew) shape. Equivalently, a skew shape is anti-straight if it can be written as λ/μ, where λ is a rectangle.

(a) Show that if X has anti-straight shape and $T \sqcup X$ makes sense, then $J^X(T)$ has anti-straight shape and depends only on T, not on X. Denote this tableau by $J^\square(T)$.

(b) Show that if T has straight shape λ, then the shape of $J^\square(T)$ is the $180°$ rotation of λ. [Hint: using dual equivalence, it suffices to do this for one specially chosen T of the given shape.]

(c) Schützenzerger's *evacuation* operator $T \to \mathrm{ev}(T)$ is defined as follows. Given T of straight shape, $\mathrm{ev}(T)$ is the tableau obtained from $J^\square(T)$ by rotating it $180°$ and renumbering the entries in reverse order, $1, 2, \ldots, n \mapsto n, \ldots, 2, 1$. Show that evacuation is an involution.

9. Let $\pi \mapsto (P(\pi), Q(\pi))$ be the Robinson-Schensted-Knuth correspondence from permutations in S_n to pairs of standard tableaux of size n.

Define $w_0 \in S_n$ by $w_0(i) = n+1-i$. Show that $(P(\pi \circ w_0), Q(\pi \circ w_0)) = (P(\pi)^*, \mathrm{ev}(Q(\pi))^*)$, where the evacuation operator $\mathrm{ev}(-)$ is as defined in the preceding problem, and T^* is the transpose of a tableau T (with shape λ^* if T has shape λ).

10. Let $K_{\lambda\mu} = \langle s_\lambda, h_\mu \rangle$ be the number of semistandard Young tableaux of shape λ and content μ. Show that $K_{\lambda\mu}$ is equal to the dimension of the space of invariants $V_\lambda^{S_\mu}$, where V_λ is the irreducible representation of S_n indexed by the partition λ, and S_μ is the Young subgroup $S_{\mu_1} \times \cdots \times S_{\mu_l} \subseteq S_n$.

11. Let V_λ be the irreducible representation of S_n whose character corresponds under the Frobenius characteristic map to the Schur function s_λ.

(a) Prove that $V_{(n-1,1)}$ is the irreducible submodule of dimension of dimension $n-1$ in the defining representation of S_n on $\mathbb{C}^n$.

(b) Prove that $V_{(n-k,1^k)}$ is isomorphic to the k-th exterior power of $V_{(n-1,1)}$.

12. The *Kronecker product* on symmetric functions is defined in terms of the power-sum basis by

$$p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda.$$

Equivalently, the symmetric functions p_λ / z_λ are orthogonal idempotents with respect to $*$.

(a) Show that $s_\mu * s_\nu$ is the Frobenius characteristic of the character of the tensor product $V_\mu \otimes V_\nu$ of the irreducible S_n modules with characters χ_μ and χ_ν, where $|\mu| = |\nu| = n$ (if $|\mu| \neq |\nu|$, then $s_\mu * s_\nu = 0$).

(b) Deduce that the Kronecker coefficients $a_{\lambda,\mu,\nu}$ defined by

$$s_\mu * s_\nu = \sum_\lambda a_{\lambda,\mu,\nu} s_\lambda$$

are non-negative integers. It is an open problem to find a general combinatorial rule for these coefficients.

(c) Show that the Kronecker coefficients are also given by

$$s_\lambda[XY] = \sum_{\mu,\nu} a_{\lambda,\mu,\nu} s_\mu[X] s_\nu[Y].$$

More generally, show that the coefficients in the expansion of $s_\lambda[X_1 \dots X_k]$ in terms of products $s_{\mu_1}[X_1] \cdots s_{\mu_1}[X_1]$ are the same as the coefficients of Schur functions s_λ in the Kronecker product $s_{\mu_1} * \cdots * s_{\mu_k}$.

(d) Show that

$$\Omega[XYZ] = \sum_{\lambda,\mu,\nu} a_{\lambda,\mu,\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z].$$

In particular, $a_{\lambda,\mu,\nu}$ is symmetric in all three indices.