Math 249—Spring 2020
Problem Set 1

1. Assume we are working in a formal power series ring $R[[x]]$ over a coefficient ring R containing $\mathbb{Q}$. Define $\exp(x)$ and $\log(1+x)$ by their usual Taylor series.

(a) Given $F(x), G(x) \in R[[x]]$ such that $F(0) = 1$, show that there is a well-defined formal power series

$$F(x)^{G(x)} \underset{\text{def}}{=} \exp(G(x) \log F(x)).$$

(b) Show that the above definition satisfies the laws of exponents $F(x)^{G(x)+H(x)} = F(x)^{G(x)} F(x)^{H(x)}$, $F(x)^{G(x)H(x)} = (F(x)^{G(x)})^{H(x)}$, $F(x)^0 = 1$, $F(x)^1 = F(x)$.

2. Generalizing the definition in the previous problem in the obvious way to formal series in more than one variable, show that (over a coefficient ring containing $\mathbb{Q}$) the Newton binomial theorem

$$(1+x)^y = \sum_n \binom{y}{n} x^n$$

holds as a formal power series identity.

3. Find the number of monotone maps $f\colon \{1, \ldots, k\} \to \{1, \ldots, n\}$, where *monotone* means $f(i) \leq f(j)$ for $i \leq j$.

4. Show that $\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} \left\langle {n-1 \atop j} \right\rangle$. Using this, evaluate $\sum_{j=0}^{k} \binom{n+j}{j}$ as a single binomial coefficient.

5. Find a simple expression for the ordinary generating function in two variables

$$\sum_{n,k \geq 0} \binom{n}{k} x^n y^k,$$

and use it to deduce the identity

$$\sum_r \binom{r}{k} x^r = \frac{x^k}{(1-x)^{k+1}}.$$

6. By expanding the right hand side in partial fractions, show directly that the generating function for Stirling numbers

$$\sum_n S(n,k) x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-k\,x)}$$

is equivalent to the explicit formula

$$S(n,k) = \frac{1}{k!} \sum_j (-1)^{k-j} \binom{k}{j} j^n.$$

7. A *perfect matching* on a set S of $2n$ elements is a partition of S into n blocks of 2 elements each. Taking $S = [2n] = \{1, 2, \cdots, 2n\}$, and thinking of the blocks in a matching as the edges of a graph, call edges of the form $\{i, i+1\}$ *short*, and all other edges *long*.

(a) Show that the number of perfect matchings on a $2n$-element set is

$$(2n-1)(2n-3)\cdots 3 \cdot 1.$$

(b) Let $M_n(x)$ be the ordinary generating function that counts perfect matchings on $[2n]$ with weight x^s, where s is the number of short edges, so for instance $M_2(x) = 1 + x + x^2$. Prove the recurrence

$$M_n(x) = (x + 2n - 2)M_{n-1}(x) + (1-x)\frac{d}{dx}M_{n-1}(x).$$

8. Let $p_+(n)$ be the the number of partitions of n with an even number of parts and $p_-(n)$ the number with an odd number of parts. Let $p_{DO}(n)$ be the number of partitions of n with distinct odd parts, and let $k(n)$ be the number of partitions λ of n such that $\lambda = \lambda^*$. Prove that

$$k(n) = p_{DO}(n) = (-1)^n(p_+(n) - p_-(n)).$$

Hint: if $\lambda = \lambda^*$, dissect the diagram of λ into 'hooks' whose sizes are odd and distinct.

9. The *Durfee square* of a partition λ is the largest $k \times k$ square that fits inside its Young diagram. Use Durfee squares to prove the following identities [Stanley, 2nd ed., Proposition 1.8.6(b) and Ex. 1.76]:

(a)

$$\prod_{i \geq 1} \frac{1}{1 - tx^i} = \sum_{k \geq 0} \frac{t^k x^{k^2}}{(1-x)\cdots(1-x^k)\,(1-tx)\cdots(1-tx^k)}$$

(b)

$$\prod_{i \geq 1}(1 + tx^{2i-1}) = \sum_{k \geq 0} \frac{t^k x^{k^2}}{(1-x^2)\cdots(1-x^{2k})}$$

10. Use Durfee squares to obtain an identity analogous to part (a) of the previous problem, but enumerating partitions with distinct parts. Then deduce *Sylvester's identity*

$$\prod_{i \geq 1}(1 - tx^i) = 1 + \sum_{n \geq 1}(-1)^n t^n \left(x^{(3n^2+n)/2} \prod_{i=1}^{n} \frac{1 - tx^i}{1 - x^i} + x^{(3n^2-n)/2} \prod_{i=1}^{n-1} \frac{1 - tx^i}{1 - q^i} \right)$$

and *Euler's pentagonal number theorem*

$$\prod_{i \geq 1}(1 - x^i) = 1 + \sum_{n \geq 1}(-1)^n (x^{(3n^2+n)/2} + x^{(3n^2-n)/2}).$$

11. Show that the Stirling numbers of the second kind $S(n,k)$ have a q-analog $S_q(n,k)$ characterized by any of the following properties, where $[k]_q = (1-q^k)/(1-q) = 1+q+\cdots+q^{k-1}$ is the q analog of k.

(a) They satisfy the recurrence

$$S_q(n,k) = [k]_q S_q(n-1,k) + q^{k-1} S_q(n-1,k-1),$$

with inital condition $S_q(0,n) = S_q(n,0) = \delta_{0,n}$.

(b) They satisfy the following q-analog of the classical formula $x^n = \sum_k S(n,k)(x)_k$:

$$[r]_q^n = \sum_k S_q(n,k)[r]_q[r-1]_q \cdots [r-k+1]_q$$

(c) For each k, they are given by the ordinary generating function

$$\sum_n S_q(n,k)x^n = \frac{q^{\binom{k}{2}} x^k}{(1-x)(1-[2]_q x)\cdots(1-[k]_q x)}.$$

(d) Given a partition $\pi = \{B_1, \ldots, B_k\}$ of $[n]$, with the blocks numbered so that $\min(B_i) < \min(B_j)$ for $i < j$, define $\nu(\pi) = \sum_i (i-1)|B_i|$. Then $S_q(n,k) = \sum_\pi q^{\nu(\pi)}$, where the sum is over partitions of $[n]$ into k blocks.

12. Given a species F, define $F'(S)$ to be the set of F structures on the set $S \coprod \{*\}$ given by adjoining a new distinguished element to S. For example, if F is the species of unrooted trees, then F' is the species of rooted forests.

(a) Show that the exponential generating function for F' is the derivative $F(x)'$ of the exponential generating function for F.

(b) Describe F' when F is the species L of linear orderings, find both exponential generating functions from other principles, and check part (a) in this case.

13. A *distribution* is a function together with a linear ordering on the preimage of each element of the codomain. Let J_k be the species such that $J_k(S)$ is the set of distributions from S to $\{1, \ldots, k\}$.

(a) Find the exponential generating function for J_k, and from it obtain the formula $(n+k-1)_n$ for the number of distributions from an n element set to a k element set.

(b) Do the same for the species of *surjective* distributions, obtaining the formula

$$(n)_k(n-1)_{n-k}$$

for the number of them.

14. Let M be the species such that $M(S)$ is the set of perfect matchings on S.

(a) Find its exponential generating function $M(x)$ directly from species principles.

(b) Deduce algebraically the formula $n!! = (2n-1)(2n-3)\cdots 3\cdot 1$ for the number of perfect matchings on a set of $2n$ elements.

In the following problems we continue to use the 'double factorial' notation $n!!$ for the product the first n odd numbers.

15. Let e_{2n} be the number of permutations σ of a set of $2n$ elements with the property that every cycle of σ has even length.

(a) Find the exponential generating function $\sum_n e_{2n} x^{2n}/(2n)!$.

(b) Deduce that $e_{2n} = (n!!)^2$.

16. (a) From the preceding problems it follows that the number of permutations of a $2n$ element set S with only even cycles is equal to the number of pairs of perfect matchings on S. Construct a direct bijection between the two (to do this you will probably need to fix the set S to be $[2n]$ and make some auxiliary choices).

(b) Show that the species of permutations with even-length cycles and the species of pairs of perfect matchings are not equivalent. This can be understood as explaining the need for auxiliary choices in part (a).

17. (a) Find the exponential generating function $\sum_n r_n x^n/n!$, where r_n is the number of permutations of odd order of an n element set.

(b) Deduce that $r_{2n} = (n!!)^2$ and $r_{2n+1} = (2n+1)(n!!)^2$.

18. A map $f\colon S \to S$ is *idempotent* if $f^2 = f$. Find the exponential generating function for the species of idempotent maps.

19. (a) The *diameter* d of a tree T is the maximum length of a path in T (a path of length n has n edges and $n+1$ vertices). Prove that if d is even then all paths of length d have the same middle vertex, called the *center* of T, and if d is odd, then all paths of length d have the same middle edge, called the *bicenter* of T.

(b) Show that if d is even, the species of labelled unrooted trees of diameter d is equivalent to the species of labelled rooted trees of height $d/2$ with the property that at least two children of the root are roots of subtrees of height $d/2-1$.

(c) Show that if d is odd, the species of labelled unrooted trees of diameter d is equivalent to the species of unordered pairs of disjoint rooted trees of height $(d-1)/2$.

(d) Let T_h be the species of labelled rooted trees of height h, and let $T_{\leq h} = T_0 + \cdots + T_h$. Show that these are given by the recurrence

$$\begin{aligned} T_0 &= X \\ T_h &= X((E-1)\circ T_{h-1})(E \circ T_{\leq h-2}) \quad \text{for } h > 0. \end{aligned}$$

(e) Use (a), (b) and (c) to express the species U_d of labelled unrooted trees of diameter d in terms the species T_h.

(f) Use (d) and (e) to calculate the number of labelled unrooted trees of diameter d on n vertices, for $n \leq 5$ and all d. Check that your answers summed over d agree with the known formula n^{n-2}.

(g) Use (d) and (e) to calculate the number of unlabelled unrooted trees of diameter d on n vertices, for $n \leq 5$ and all d. Check your answers by listing the trees.

20. Use species generating function methods to find the number of unlabelled unrooted forests on n vertices for $n \le 6$.

21. (a) Show that the exponential generating function for the species of labelled unrooted forests is given by

$$F(x) = \exp \sum_{n=1}^{\infty} n^{n-2} \frac{x^n}{n!}.$$

(b) Use (a) to calculate the number of labelled unrooted forests on $[n]$ for $n \le 6$.

(c) Modify (a) to get a mixed ordinary/exponential generating function $F(t, x)$ for the species of labelled unrooted forests weighted by t^k for a forest with k components. Use this to refine your answer to (b) to count forests by number of components.

22. (a) Verify by direct calculation that the cycle index Z_C for the species of cyclic orderings (i.e., permutations with one cycle) is given by

$$Z_C = \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \frac{1}{1 - p_n},$$

where ϕ is Euler's totient function: $\phi(n)$ is the number of integers $r \in [n]$ relatively prime to n.

(b) Check that $Z_C(x, 0, \ldots)$ and $Z_C(x, x^2, \ldots)$ agree, respectively, with the exponential generating function for the species of cyclic orderings, and the ordinary generating function for cyclic orderings up to isomorphism.

(c) Recall that the cycle index of the trivial species is given by

$$Z_E = \exp \sum_{n=1}^{\infty} p_n/n.$$

Verify that the plethysm $Z_E * Z_C$ agrees with the formula we obtained by direct calculation for the cycle index of the species of permutations,

$$Z_P = \prod_{n=1}^{\infty} \frac{1}{1 - p_n}.$$

23. (a) From Cayley's tree generating function derive the identity

$$\sum_F \prod_{i=1}^{n} x_i^{c_F(i)} = \binom{n-1}{k-1} (x_1 + \cdots + x_n)^{n-k},$$

where the sum is over rooted forests F with k components on vertices $\{1, \ldots, n\}$, and $c_F(i)$ denotes the number of children of vertex i in F.

(b) Deduce the identity

$$\sum_F \prod_{i=1}^{n} h_{c_F(i)} = \binom{n-1}{k-1} \langle \frac{x^{n-k}}{(n-k)!} \rangle H(x)^n,$$

where $H(x) = \sum_m h_m x^m/m!$ is a generic formal power series written in exponential form and $\langle \cdot \rangle$ denotes taking a coefficient.

(c) Let H be the 'generic species' with exponential generating function $H(x)$, that is, the trivial species, but enumerated by assigning weight h_n to the one structure on any set with n elements. Let $F(x)$ be the solution of the formal functional equation

$$F(x) = x\, H(F(x))$$

(that is, assuming h_0 invertible, the functional composition inverse of $x/H(x)$). Using (b) and a species interpretation of $F(x)$, obtain the generalized Lagrange inversion formula

$$\langle \frac{x^n}{n!} \rangle F(x)^k/k! = \binom{n-1}{k-1} \langle \frac{x^{n-k}}{(n-k)!} \rangle H(x)^n,$$

or equivalently,

$$\langle x^n \rangle F(x)^k = \frac{k}{n} \langle x^{n-k} \rangle H(x)^n.$$

24. In class we showed that the number of unlabelled ordered rooted trees with $n+1$ vertices is equal to the Catalan number $C_n = \binom{2n}{n}/(n+1)$.

(a) A *binary tree* is an ordered rooted tree in which every non-leaf node has exactly two children. Note that every binary tree has an odd number of vertices. Show that the number of binary trees with $2n+1$ vertices is equal to C_n.

(b) A *lisp tree* is an at most binary tree in which the children of each node are distinguished as *left* and *right*. That is, if there are two children, they are ordered, and if there is one child, we still distinguish two cases. Show that the number of lisp trees with n vertices is equal to C_n.

(c) Find bijections between the above mentioned three kinds of trees enumerated by C_n.

25. Prove that the number of ordered rooted trees with $n+1$ vertices and j leaves is equal to

$$\frac{1}{n+1} \binom{n+1}{j} \binom{n-1}{n-j}.$$

26. In class we saw that if formal series $A(x)$ and $B(x)$ are related by $xB(x)$ being the functional composition inverse of $x/A(x)$, then the coefficient b_n of x^n in $B(x)$ is given in terms of the coefficients of $A(x)$ by summing the weight

$$\prod_{v \in T} a_{c_T(v)}$$

over ordered unlabelled rooted trees T on $n+1$ vertices, where $c_T(v)$ is the number of children of vertex v in T.

We also saw, separately, that b_n is the sum, over Dyck paths D of order n, of the monomial $a_{r_0}, \ldots, a_{r_n}$, where r_i is the number of South steps in D on the vertical line $x = i$ (so $r_n = 0$ always).

Find a bijection between Dyck paths D of order n and unlabelled ordered rooted trees T on $n+1$ vertices such that the monomial $\prod_{v \in T} a_{c_T(v)}$ for T is the same as the monomial $a_{r_0}, \ldots, a_{r_n}$ for D.

(Both sets have size equal to the Catalan number C_n. The simple bijection between them that we discussed in class doesn't match up the monomials.)