Math 249, Monday, April 13

Classical LR rule Yamanouchi word w : every tail of w has $\#1\text{'s} \geq \#2\text{'s} \geq \cdots$

Tableau $T \in SSYT(\nu/\mu)$ Yamanouchi $\overset{}{\underset{\text{def}}{\iff}}$ reading word $w(T)$ is Yam.

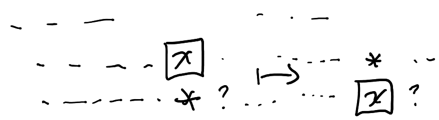
Lemma 1 Slides preserve Yam. Tableaux

Pf. · (forward slides, reverse are similar)

· Claim: as hole moves, tableau remains Yam.

· Enough to look 1-2 tableaux

$* \boxed{x} \quad \boxed{x}\, *$ doesn't change row reading word



w changes $\;- x\, y - \;\mapsto\; - y\, x -$

If $x = 1$, change in w can't destroy Yamanouchi property.

$x = 2$

$$\begin{matrix} 1 \cdots 1\, \overbrace{2 \cdots 2}^{2}\, 2\, \boxed{\ } \\ \underbrace{\cdots 1\, 1}_{\substack{y \\ k\ 1\text{'s}}}\, * \end{matrix} \longmapsto \begin{matrix} 1 \cdots 1\ 2 \cdots 2 \\ 1 \cdots 1\, 2 \end{matrix}$$

w changes : $\;-\; 2^k\, \underline{2}\, 1^k \underbrace{\;-\;}_{z} \;\mapsto\; -\; 2^k\, 1^k\, 2 \underbrace{\;-\;}_{z}$

Yam. $\Rightarrow$ z has $\#1\text{'s} > \#2\text{'s}$ $\Rightarrow$ new word is Yam.

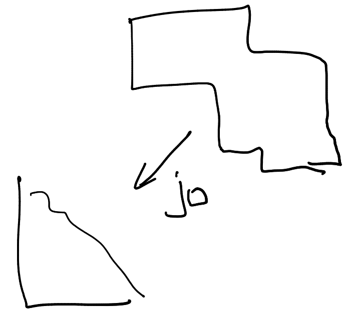
Lemma 2 Straight shape λ has unique Yam. $T \in SSYT(\lambda)$

$$\begin{matrix} 4 & & & \\ 3 & 3 & 3 & \\ 2 & 2 & 2 & \\ 1 & 1 & 1 & 1 \end{matrix}$$

lemma 1 + lemma 2 + Jeu-de-Taquin

$$\Rightarrow \underbrace{c^{\lambda}_{\nu/\mu} = |\,\mathrm{Yam}(\nu/\mu, \lambda)\,|}$$

Theorem ↑ Classical LR rule.



Robinson-Schensted-Knuth (RSK) Bijection

Permutations $\longleftrightarrow$ pairs (P, Q) of SYT of same shape

2 6 1 3 4 7 5

$$P \quad \emptyset \xleftarrow{} 2 \qquad 2 \xleftarrow{} 6 \qquad 26 \xleftarrow{} 1 \qquad \begin{matrix} 2 \\ 1\,6 \end{matrix} \xleftarrow{} 3 \qquad \begin{matrix} 2\,6 \\ 1\,3 \end{matrix} \xleftarrow{} 4 \qquad \begin{matrix} 2\,6 \\ 1\,3\,4 \end{matrix} \xleftarrow{} 7 \qquad \begin{matrix} 2\,6 \\ 1\,3\,4\,7 \end{matrix} \xleftarrow{} 5 \longrightarrow \begin{matrix} 2\,6\,7 \\ 1\,3\,4\,5 \end{matrix}$$

$$\left(\begin{matrix} 6 \\ 2\,3\,7 \\ 1\,2.5\,4\,5 \end{matrix} \xleftarrow{} 2.5 \right)$$

$$Q \quad \emptyset \qquad 1 \qquad 1\,2 \qquad \begin{matrix} 3 \\ 1\,2 \end{matrix} \qquad \begin{matrix} 3\,4 \\ 1\,2 \end{matrix} \qquad \begin{matrix} 3\,4 \\ 1\,2\,5 \end{matrix} \qquad \begin{matrix} 3\,4 \\ 1\,2\,5\,6 \end{matrix} \qquad \begin{matrix} 3\,4\,7 \\ 1\,2\,5\,6 \end{matrix}$$

This is invertible.

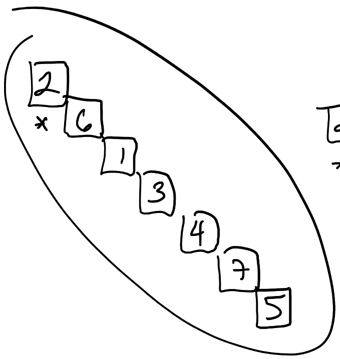
$$\begin{matrix} 2 \\ 1\,6 \end{matrix} \uparrow 3 \qquad \begin{matrix} 2\,6 \\ 1\,3 \end{matrix} \uparrow 4 \qquad \begin{matrix} 2\,6 \\ 1\,3\,4 \end{matrix} \uparrow 7 \qquad \begin{matrix} 2\,6 \\ 1\,3\,4\,7 \end{matrix} \uparrow 5 \qquad \begin{matrix} 2\,6\,7 \\ 1\,3\,4\,5 \end{matrix}$$

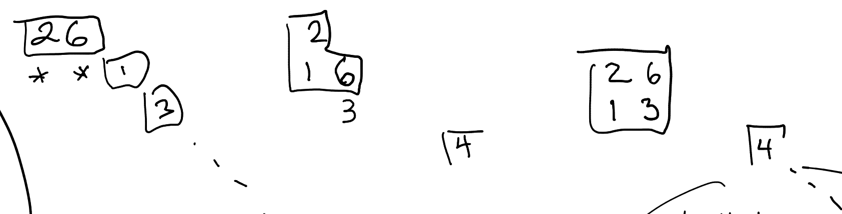
$$\begin{matrix} 11\ 13 \\ 6\ 7 \\ 3\ 5\ 10 \\ 1\ 2\ 8\ 9\ 12 \end{matrix} \uparrow 4 \qquad \begin{matrix} 11\ 13 \\ 6\ 7\ 10 \\ 3\ 5\ 8 \\ 1\ 2\ 4\ 9\ 12 \end{matrix}$$

$\pi = 2613475$

$P(\pi)$

P: $\emptyset \xleftarrow{} 2$, 2 ← 6, 26 ← 1, $\begin{matrix}2\\16\end{matrix}$ ← 3, $\begin{matrix}26\\13\end{matrix}$ ← 4, $\begin{matrix}26\\134\end{matrix}$ ← 7, $\begin{matrix}26\\1347\end{matrix}$ ← 5 → $\begin{matrix}267\\1345\end{matrix}$





10 11 12

2 6 9

1 3 4 7 8 13 ← 5

10 11 12

2 6 9 7

1 3 4 5 8 13

10 11 12

2 6 7

1 3 4 5 8 13

10

9 11 12

2 6 9

7

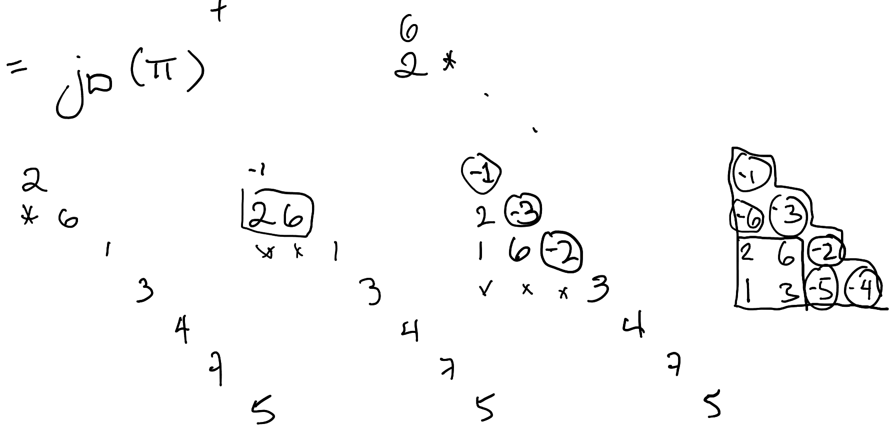
1 3 4 7 8 13 (5)

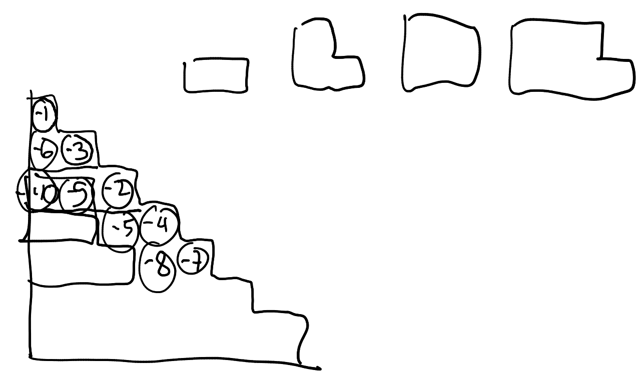
7

1 3 4 5 8 13

(7) 1 3 4 5 8 13

RSK $P(\pi) = j_D(\pi)$





V: jdt(π) determines $Q(\pi)$, i.e. $\sim$ class of π determines $Q(\pi)$

$Q(\pi)$ is constant on $\sim$ equiv. classes.

(P, Q) determine π : $\#\ \pi$ with given $Q(\pi) = Q$ is $\leq |SYT(\lambda)|$

where $\lambda = \text{shape}(Q)$

If $\lambda = \text{sh jdt}(\pi)$, then jdt : $(\sim \text{class of } \pi) \to SYT(\lambda)$

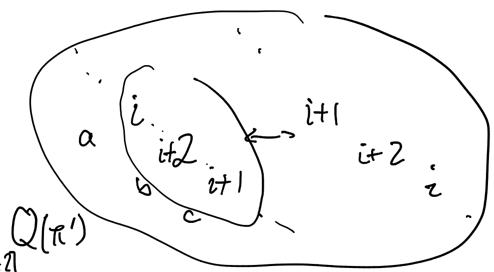
$$\Rightarrow \sim \text{equiv of } \pi = \{\text{all } \pi' \text{ with } Q(\pi') = Q\}$$

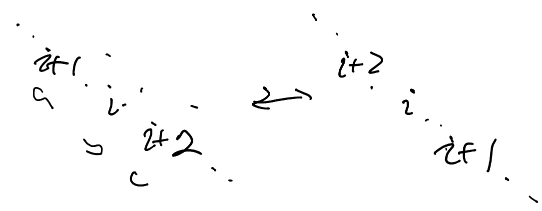
$$Q(\pi) = Q(\pi') \iff \pi \sim \pi'$$

$\pi \underset{[i,i+2]}{\sim} \pi' \longrightarrow Q(\pi) = Q(\pi')$

$\downarrow$

$P(\pi) \underset{[i,i+2]}{\sim} P(\pi')$





$\pi^{-1} \underset{[i,i+2]}{\sim} (\pi')^{-1} \longrightarrow Q(\pi) \underset{[i,i+1,i+2]}{\sim} Q(\pi')$

$\downarrow$

$P(\pi) = P(\pi')$

—— a c b ——   —— b a c ——

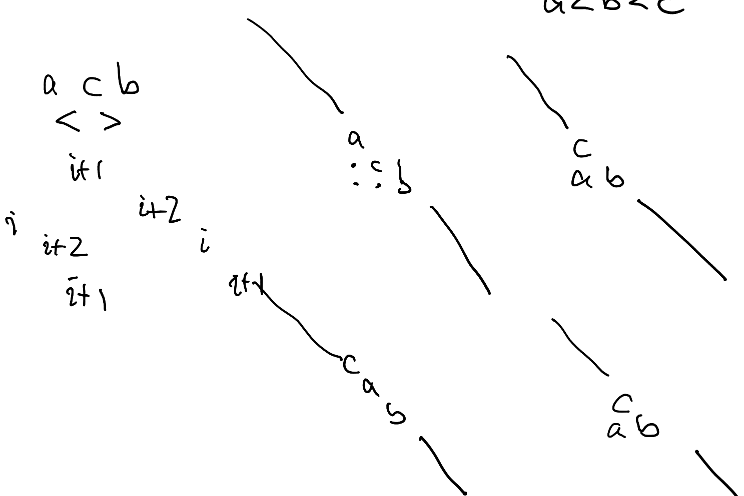
$\updownarrow$

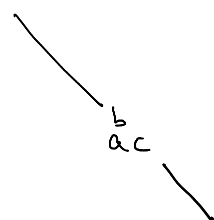
—— c a b ——   —— b c a ——

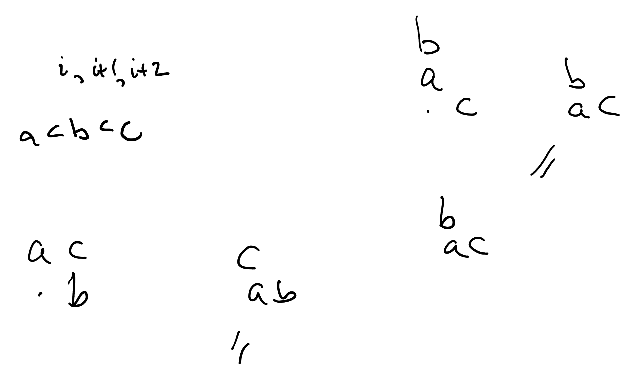
Knuth relations

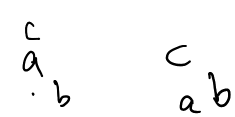
$i, i+1, i+2$ $i, i+1, i+2$

$a<b<c$ $a<b<c$









by an $\underset{[i,i+1,i+2]}{\sim}$

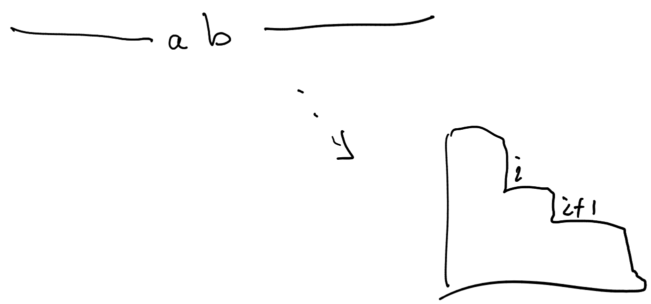
$\downarrow$

$Q(\pi)$ and $Q(\pi')$ differ only in entries i, $i+1$, $i+2$

• $D(Q(\pi)) = D(\pi^{-1})$ (= descent of π as a word)

$4\,1\,3\,2$

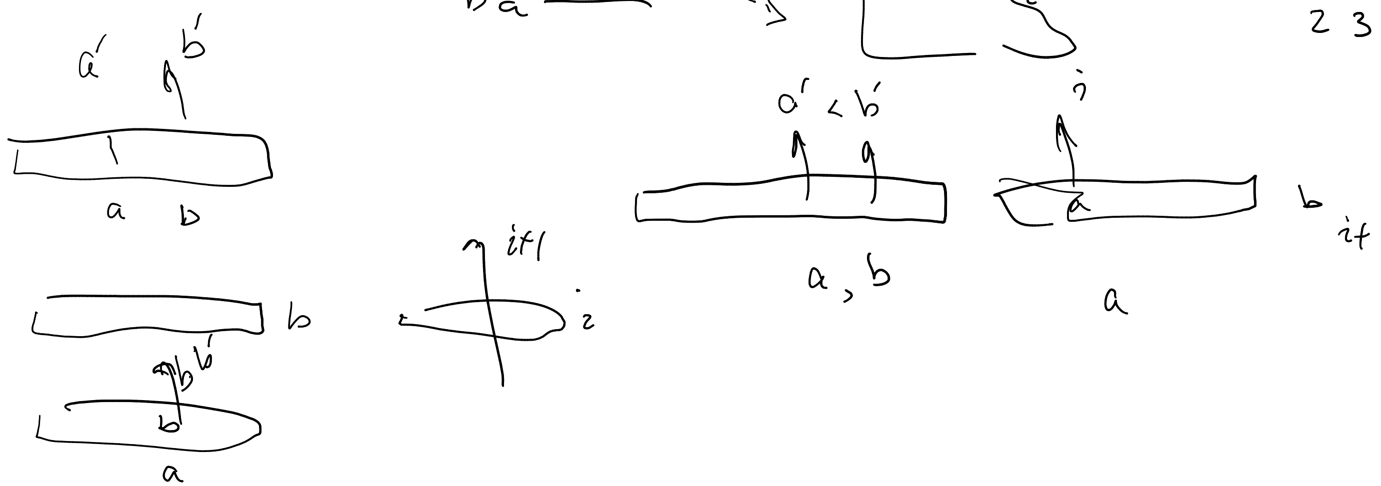
$\underline{1}$ $\underline{3}$



4
1
3
2

$D = \{2, 3\}$

$(4\,1\,3\,2)^{-1}$
$=$
$2\,4\,3\,1$
$<\ >\ >$
$2\ 3$



will imply $\pi \to (P, Q)$

$\pi^{-1} \to (Q, P)$

(next time)