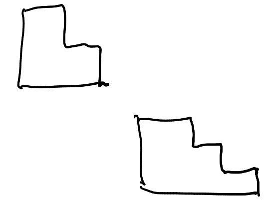
Math 249, Monday April 6

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu}\, s_\nu$$

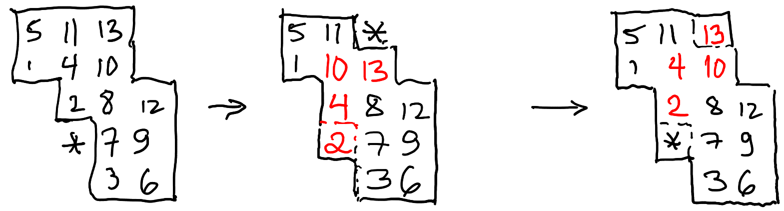
$$s_{\nu/\mu} = \sum_\lambda c^\lambda_{\nu/\mu}\, s_\lambda$$



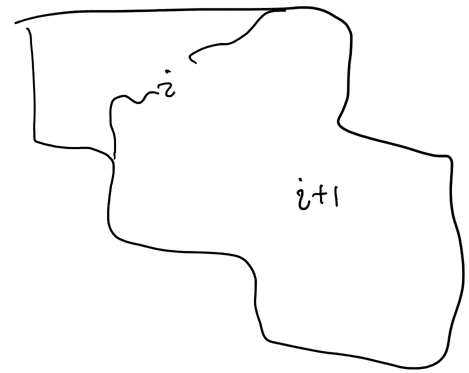
$$s_{\lambda \oplus \mu} = s_\lambda \cdot s_\mu$$

$$\langle s_\lambda, s_{\nu/\mu} \rangle = \langle s_\lambda s_\mu, s_\nu \rangle$$

Jeu-de-taquin : slides



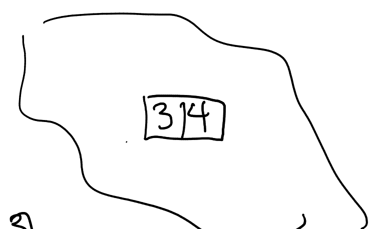
forward slide — reverse slide — undo each other



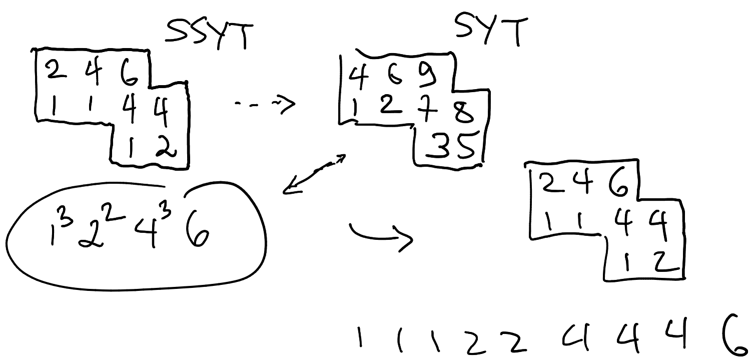
Descent set of a standard tableau : $i{+}1$ i

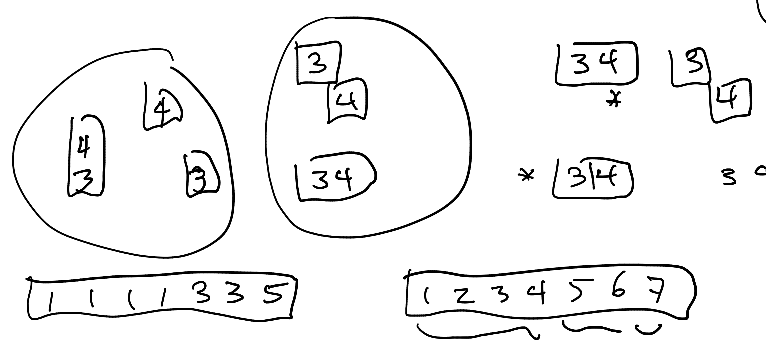
$D(T) = \{3, 4, 6, 7, 9, 10, 12\}$ preserved by slides !

(check shapes of size 2)



Standardization :





Standard tableau T, list of letters $1^{\alpha_1} 2^{\alpha_2} \cdots$ are compatible iff
$|T| = |\alpha|$ and $D(T) \subseteq \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{k-1}\} = D(\alpha)$

Descent Sets $\longleftrightarrow$ compositions of n

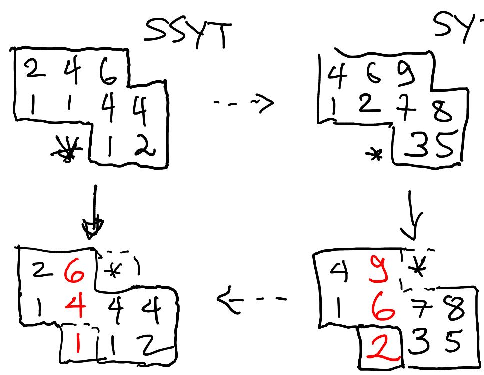
(for $|T|=n$) $\qquad \alpha_1, \ldots, \alpha_k \qquad \alpha_1+\cdots+\alpha_k = n$

$\subseteq \{1,\ldots,n-1\} \qquad \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{k-1}\} \subseteq \{1,\ldots,n-1\}$

$D \longmapsto \alpha(D)$ $\qquad \{2,5\}$

$D(\alpha) \longleftarrow \alpha$ $\qquad \{2,3,2\}$

Extends jeu-de-taquin to SSYT:



$\to 1\,1\,1\,2\,2\,4\,4\,4\,6$

Gessel's Fundamental Quasi-symmetric functions

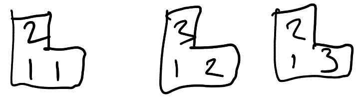
$$Q_\alpha(x_1, x_2, \ldots) \underset{\text{def}}{=} \sum_{\substack{a_1 \le \cdots \le a_n \\ a_i < a_{i+1} \\ \text{if } i \in D(\alpha)}} x_{a_1} \cdots x_{a_n} \qquad n = |\alpha|$$

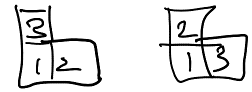
$\{2\}$

Ex. $Q_{(n)} = h_n(x_1, x_2, \ldots)$ $\qquad Q_{(1^n)} = e_n(x_1, x_2, \ldots)$ $\qquad Q_{(2,1)} = \sum_{a_1 \le a_2 < a_3} x_{a_1} x_{a_2} x_{a_3}$

$\quad s_{(n)}\ \boxed{1\ 2\ \cdots\ n} \qquad s_{(1^n)}\ \begin{array}{|c|}\hline n \\ \vdots \\ 2 \\ 1 \\ \hline\end{array}$ $\qquad = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2 x_3 + \cdots$

$$s_\nu = \sum_{T \in SYT(\nu)} Q_{\alpha(D(T))}(x)$$

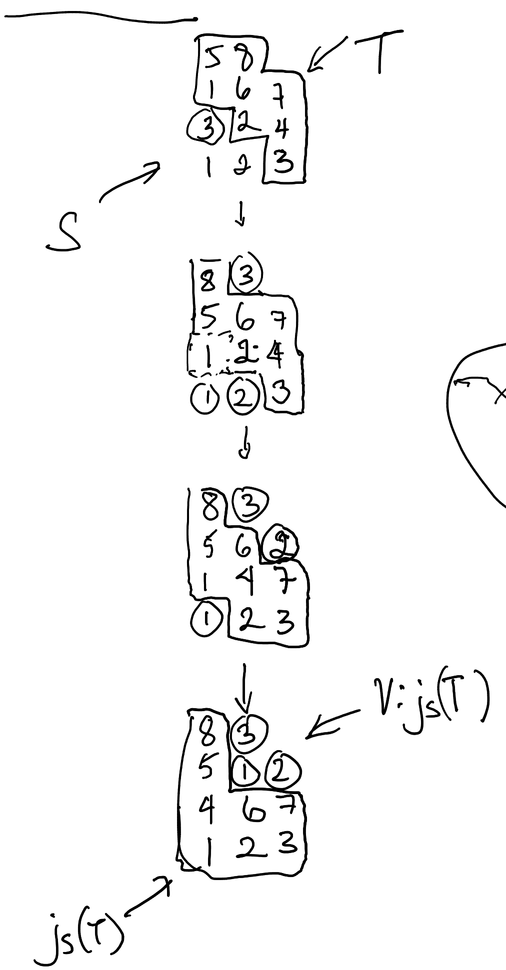




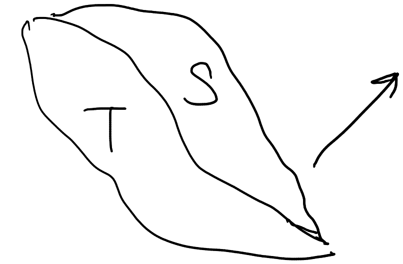
$D = \{2\}$ $\qquad$ $D = \{1\}$

$\alpha = 2,1$ $\qquad$ $\alpha = 1,2$

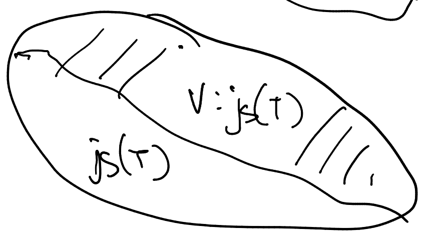
$$s_{21} = Q_{21} + Q_{12} = m_{21} + 2m_{111}$$

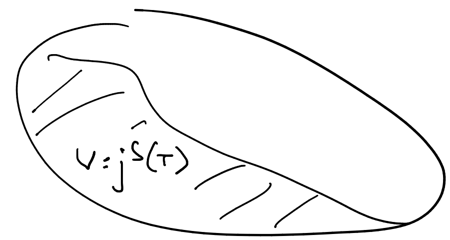


slides on T in order given by S $=$ $js(T)$

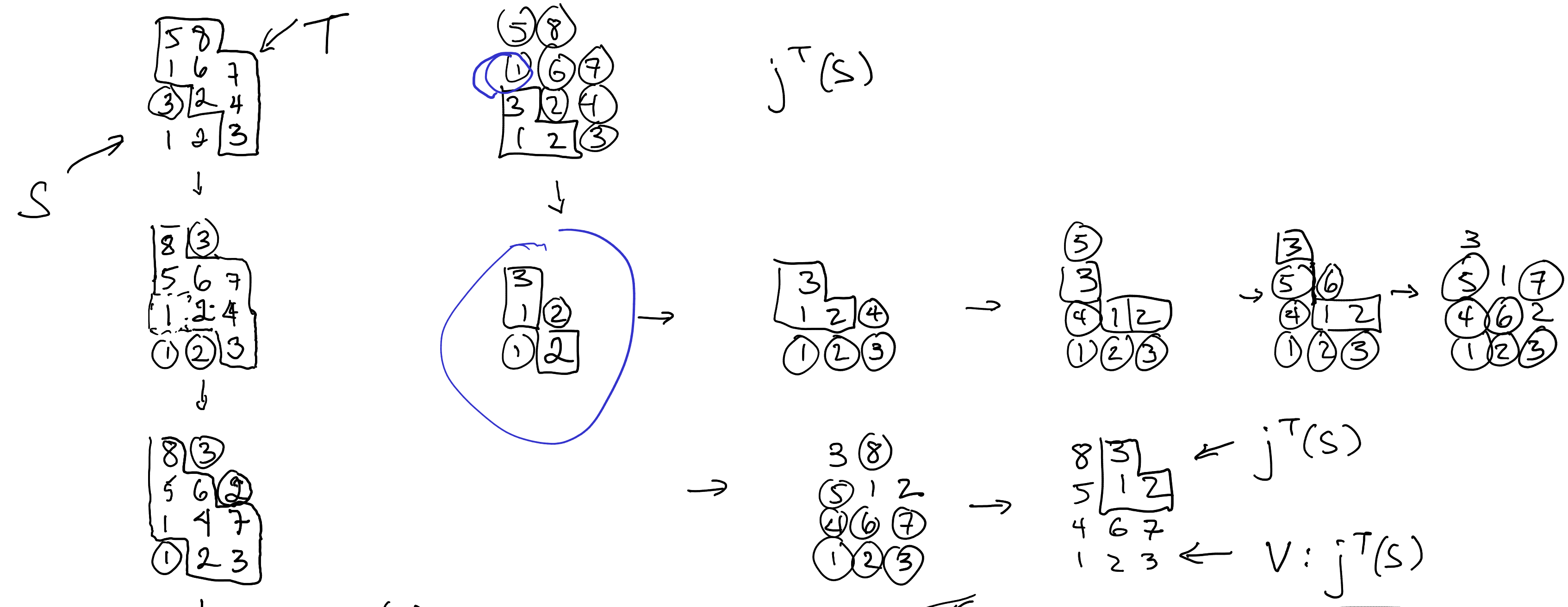


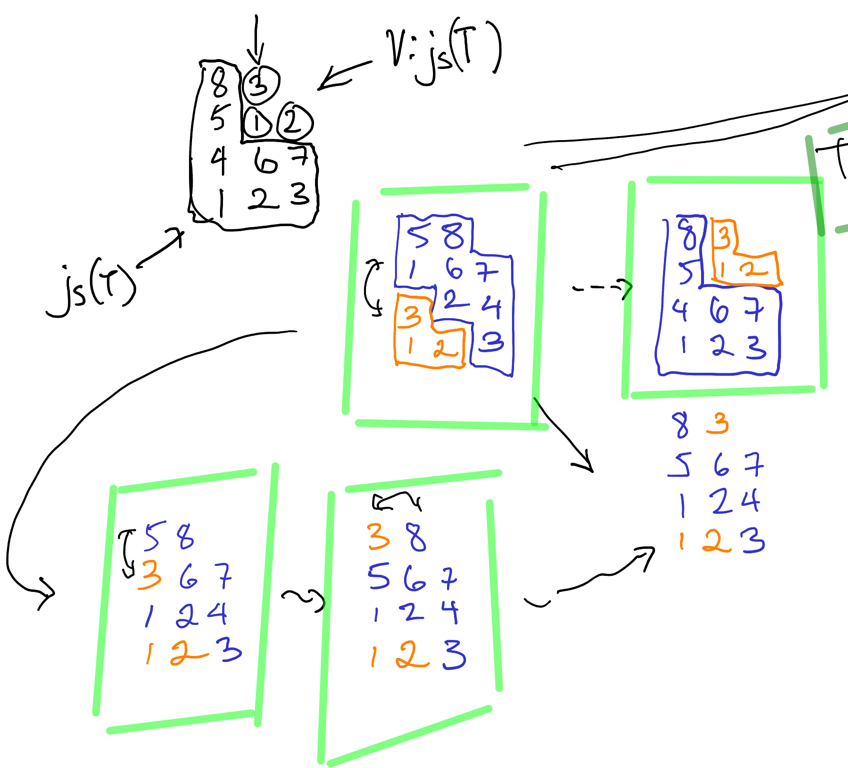
$j^S(T)$





$$j^{V:js(T)}(js(T)) = T \quad \text{(clear)}$$





Theorem

$$j^T(S) = V: js(T)$$

$$js(T) = V: j^T(S)$$

1 2 3 < 1 2 3 4 5 6 7 8

1 2 1 3 2 3 4 5 6 7 8

1 2 1 2 3 4 3 5 6 7 8

1 2 1 2 3 4 5 3 6 7 8

1 2 1 2 3 4 5 6 7 8 3

1 1 ··· 8 2 3

1 2 ··· 8 1 2 3