Math 249 Problem Set 5

This problem set is due Friday, May 15. You can turn it in either by e-mail or at my office, 855 Evans Hall - slip it under the door if you come by when I am not there.

1. Recall from class that the Schur functions $s_\lambda(x_1, \ldots, x_n)$ are the characters of the irreducible polynomial representations of $GL_n(\mathbb{C})$. Let $\rho \colon : GL_n \to GL_m$ be an irreducible polynomial representation of GL_n on $\mathbb{C}^m$ and $\sigma \colon GL_m \to GL_l$ an irreducible polynomial representation of GL_m on $\mathbb{C}^l$. Then $\sigma \circ \rho$ is a polynomial representation of GL_n on $\mathbb{C}^l$.

Show that if the character of ρ is $s_\lambda(x_1, \ldots, x_n)$ and the character of σ is $s_\mu(x_1, \ldots, x_m)$, then the character of $\sigma \circ \rho$ is the plethysm $s_\mu[s_\lambda[x_1 + \cdots + x_n]]$. Deduce that the coefficients $d^\lambda_{\mu\nu}$ defined by

$$s_\mu[s_\nu[X]] = \sum_\lambda d^\lambda_{\mu\nu} s_\lambda[X],$$

are non-negative integers. It is an open problem to find a combinatorial rule for the computation of plethysm coefficients in general, although some special cases are known.

2. Prove the formula for complete homogeneous symmetric functions in terms of elementary symmetric functions

$$h_n = \sum_{|\lambda|=n} (-1)^{n-l(\lambda)} \binom{l(\lambda)}{r_1, r_2, \ldots, r_k} e_\lambda,$$

where $\lambda = (1^{r_1}, 2^{r_2}, \ldots, k^{r_k})$.

3. Express $m_\lambda(1, 1, \ldots, 1)$, with n ones, as a more familiar combinatorial quantity.

4. Let ∂p_k be the operator on symmetric functions given by partial differentiation with respect to p_k, under the identification of the algebra of symmetric functions with the polynomial ring $\mathbb{Q}[p_1, p_2, \ldots]$. Show that ∂p_k is adjoint with respect to the Hall inner product to the operator of multiplication by p_k/k.

5. (a) Show that the coefficient of $m_\lambda[X]m_\nu[Y]$ in $m_\nu[X+Y]$ is equal to 1 if $\nu = \lambda \cup \mu$, and zero otherwise.

(b) Use part (a) and the fact that $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$ to show that the plethystic substitution $X \to X + Y$ is adjoint to multiplication, in the sense that for all symmetric polynomials f, g, h we have

$$\langle f, gh \rangle = \langle f[X+Y], g[X]h[Y] \rangle_{XY},$$

where $\langle -, - \rangle_{XY}$ is the inner product on $\Lambda(X) \otimes \Lambda(Y)$ defined so that $\{m_\lambda[X]m_\nu[Y]\}$ and $\{h_\lambda[X]h_\nu[Y]\}$ are dual bases.

6. Let ϵ be a fictitious alphabet such that $p_k[\epsilon] = \delta_{1,k}$. Stated more correctly, this means we are to interpret $f[\epsilon]$ as the image of f under the homomorphism $\Lambda \to \mathbb{Q}$ mapping p_k to $\delta_{1,k}$.

(a) Prove the identity $f[\epsilon] = \langle f, \exp(p_1) \rangle$.

(b) Prove the identity $f[\epsilon] = \lim_{n\to\infty}(f[nx])_{x\mapsto 1/n}$.

(c) Show that $e_k[\epsilon] = h_k[\epsilon] = 1/n!$.

(d) More generally, show that $s_\lambda[\epsilon] = f_\lambda/n!$, where $|\lambda| = n$ and f_λ is the number of standard Young tableaux of shape λ.

7. [from I. G. Macdonald, Symmetric Functions and Hall Polynomials]

(a) Recall from class that $h_n = \sum_{|\lambda|=n} p_\lambda/z_\lambda$, where $z_\lambda = \prod_i i^{r_i} r_i!$ for $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$. Show that this is equivalent to Newton's determinant formula

$$h_n = \frac{1}{n!}\det\begin{bmatrix} p_1 & -1 & 0 & \ldots & 0 \\ p_2 & p_1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n-1} & p_{n-2} & \cdot & \ldots & -(n-1) \\ p_n & p_{n-1} & \cdot & \ldots & p_1 \end{bmatrix}$$

(b) Show that e_n is given by the same determinant without the minus signs.

8. [also from Macdonald] Prove the identity $s_{(n-1,n-2,\ldots,1)}(x_1,\ldots,x_n) = \prod_{1\le i<j\le n}(x_i+x_j)$.

9. Prove that standard tableaux S and T are dual equivalent if and only if there exist standard tableaux S' and T' of some straight shape λ and a tableau X (of a shape ν for which $\lambda \sqcup \nu$ makes sense) such that $J^X(S') = S$ and $J^X(T') = T$.

10. Let $\lambda = (k, 1^l)$ be a hook shape. Prove the following rule for computing $s_\lambda s_\mu$, which generalizes the Pieri rules for $s_{(k)}s_\mu$ and $s_{(1^k)}s_\mu$:

(i) s_ν occurs with non-zero coefficient in $s_\lambda s_\mu$ only if ν/μ is a disjoint union of ribbons (connected skew shapes containing no 2×2 rectangle) of total size $k+l$,

(ii) in that case, the coefficient is $\binom{r-1}{h-l-1}$, where μ/ν consists of r ribbons, and h is the sum of their heights (interpreting $\binom{r-1}{h-l-1}$ as zero if $h-l-1$ is negative).

11. Compute the character table of S_5.

12. Let $K_{\lambda\mu} = \langle s_\lambda, h_\mu\rangle$ be the number of semistandard Young tableaux of shape λ and content μ. Show that $K_{\lambda\mu}$ is equal to the dimension of the space of invariants $V_\lambda^{S_\mu}$, where V_λ is the irreducible representation of S_n indexed by the partition λ, and S_μ is the Young subgroup $S_{\mu_1}\times\cdots\times S_{\mu_l} \subseteq S_n$.

13. Prove that the Frobenius characteristic map is given by

$$F\chi_V = \sum_\mu \dim(V^{S_\mu}) m_\mu,$$

where V is an S_n module and S_μ is as in the preceding problem.