Math 249 Problem Set 4

From Stanley (Vol. I 2nd ed): Exercise 3.66

Additional problems.

1. In class I sketched out some of this problem and the next. Now I ask you to fill in the details, and compute some terms by hand of generating functions we calculated in class by computer.

(a) The *diameter* d of a tree T is the maximum length of a path in T (a path with n vertices has length $n-1$). Prove that if d is even then all paths of length d have the same middle vertex, called the *center* of T, and if d is odd, then all paths of length d have the same middle edge; the path of length 1 consisting of this edge and its two endpoints is called the *bicenter* of T.

(b) Show that if d is even, the species of labelled unrooted trees of diameter d is equivalent to the species of labelled rooted trees of height $d/2$ with the property that at least two children of the root are roots of subtrees of height $d/2-1$.

(c) Show that if d is odd, the species of labelled unrooted trees of diameter d is equivalent to the species of unordered pairs of disjoint rooted trees of height $(d-1)/2$.

(d) Let T_h be the species of labelled rooted trees of height h, and let $T_{\leq h} = T_0 + \cdots + T_h$. Show that these are given by the recurrence

$$\begin{aligned} T_0 &= X \\ T_h &= X((E-1)\circ T_{h-1})(E \circ T_{\leq h-2}) \quad \text{for } h > 0. \end{aligned}$$

(e) Using products and composition of species, express the species of labelled unrooted trees of diameter d in terms the species T_h for various h.

(f) Use the above to calculate the number of labelled unrooted trees of diameter d on n vertices, for $n \leq 5$ and all d. Check your answer by summing over d.

2. Find a recurrence giving the ordinary generating function $Z_{T_h}[x]$ for unlabelled rooted trees of height h, and find an expression in terms of these for the ordinary generating functions $Z_{U_d}[x]$ for unlabelled unrooted trees of diameter d. From this, calculate the number of unlabelled unrooted trees of diameter d on n vertices for $n \leq 5$ and all d.

3. Use species generating function methods to find the number of unlabelled unrooted forests on n vertices for $n \leq 6$.

4. (a) Verify by direct calculation that the cycle index Z_C for the species of cyclic orderings (i.e., permutations with one cycle) is given by

$$Z_C = \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \frac{1}{1-p_n},$$

where ϕ is Euler's totient function: $\phi(n)$ is the number of integers $r \in [n]$ relatively prime to n.

(b) Check that $Z_C(x, 0, \ldots)$ and $Z_C(x, x^2, \ldots)$ agree, respectively, with the exponential generating function for the species of cyclic orderings, and the ordinary generating function for cyclic orderings up to isomorphism.

5. Recall that the cycle index of the trivial species is given by

$$Z_E = \exp \sum_{n=1}^{\infty} p_n/n.$$

Verify that the plethysm $Z_E * Z_C$ agrees with the formula we obtained by direct calculation for the cycle index of the species of permutations,

$$Z_P = \prod_{n=1}^{\infty} \frac{1}{1 - p_n}.$$

6. (a) Find an explicit formula for the cycle index Z_I of the species of involutions, $I(S) = \{\text{involutions } \sigma \colon S \to S\}$.

(b) Evaluate $Z_I[x]$ and verify that it agrees with the obvious ordinary generating function counting involutions up to conjugacy.

7. Denote the category of finite posets and monotone maps by $\mathbf{PoSet}_{\text{fin}}$, and the category of finite distributive lattices and $(\mathbf{0}, \mathbf{1})$-preserving lattice homomorphisms by $\mathbf{DLat}_{\text{fin}}$. The Fundamental Theorem (in a stronger form than Stanley, Theorem 3.4.1) says that the functor $J \colon \mathbf{PoSet}_{\text{fin}}^{\text{op}} \to \mathbf{DLat}_{\text{fin}}$ is an equivalence, with inverse functor $\mathcal{P} \colon \mathbf{DLat}_{\text{fin}} \to \mathbf{PoSet}_{\text{fin}}^{\text{op}}$ such that $\mathcal{P}(L)$ is the poset of join-irreducible elements of L.

In class we constructed the isomorphisms $P \cong \mathcal{P}(J(P))$ and $L \cong J(\mathcal{P}(L))$. One should also verify that these isomorphisms are functorial, although this is more or less clear from the canonical nature of their construction. Complete the proof by constructing, for a given $(\mathbf{0}, \mathbf{1})$-homomorphism $\phi \colon L \to M$ between finite distributive lattices, the unique monotone map $\psi \colon \mathcal{P}(M) \to \mathcal{P}(L)$ such that, under the identifications $L = J(\mathcal{P}(L))$ and $M = J(\mathcal{P}(M))$, we have $J(\psi) = \phi$. In particular, this construction defines how $\mathcal{P}$ is a functor.

8. The *free* **0**-**1** *distributive lattice* F_n on n generators $x_1, \ldots, x_n$ is characterized by the universal property that for every distributive lattice L with $\mathbf{0}$ and $\mathbf{1}$, and elements $a_1, \ldots, a_n \in L$, there is a unique **0**-**1**-lattice homomorphism $\phi \colon F_n \to L$ such that $\phi(x_i) = a_i$ for all i. This is a special case of the notion of free object in any variety of universal algebras, of which other examples are free groups, free modules, and polynomial rings. Prove that F_n is finite, and that in fact, $F_n \cong J(B_n)$. (See Stanley, Ex. 3.71 for a more general construction.)

9. (a) Given a finite lattice L, define $I(L)$ to be the poset of all intervals $[x, y] \subseteq L$, plus the empty set, ordered by containment. Show that $I(L)$ is a lattice and describe its meet and join.

(b) Show that the Möbius function in $I(L)$ is given by

$$\begin{aligned}&\mu([x,y],[w,z]) = \mu_L(w,x)\mu_L(y,z) \quad \text{for nonempty } [x,y],\\ &\mu(\emptyset,[w,z]) = -\mu_L(w,z).\end{aligned}$$

(c) Prove *Crapo's lemma*: let X be a subset of L, and let n_k be the number of k-element subsets of X with join equal to $\mathbf{1}$ and meet equal to $\mathbf{0}$. Then

$$\sum_k (-1)^k n_k = -\mu(\mathbf{0},\mathbf{1}) + \sum_{\substack{x\le y\\ [x,y]\cap X=\emptyset}} \mu(\mathbf{0},x)\mu(y,\mathbf{1}).$$

(d) Prove the identity in Stanley, Exercise 3.92, known as *Crapo's complementation theorem.*

10. Let $\mathcal{P}$ be the class of all finite posets which are isomorphic to products of chains. Let $C_k = [k+1]$ be the chain of length k, and for every partition λ, put $P_\lambda = C_{\lambda_1} \times \cdots \times C_{\lambda_l}$. Let $I(\mathcal{P},\cong)$ be the reduced incidence algebra of functions constant on isomorphism classes in $\mathcal{P}$, and let $\widehat{\Lambda}$ be the algebra of symmetric formal series. Define a map

$$\phi\colon I(\mathcal{P},\cong) \to \widehat{\Lambda}$$

by $\phi f = \sum_\lambda f(P_\lambda) m_\lambda$. Show that ϕ is an algebra isomorphism, and identify $\phi(\zeta)$ and $\phi(\mu)$. Which elements of $I(\mathcal{P},\cong)$ correspond to the elementary symmetric functions e_n, the complete homogeneous symmetric functions h_n, and the power-sums p_n?