Math 249 Problem Set 3

Problems from Stanley (Volume 2).

5.11, 5.13(a,b), 5.20, 5.23

Additional problems.

1. Prove that the Eulerian polynomials $A_n(x)$ satisfy the the following more symmetrical recurrence than the one in the proof of Stanley, Prop. 1.4.4:

$$A_n(x) = xA'_{n-1}(x) + x^n A'_{n-1}(x^{-1}).$$

Use this to compute $A_n(x)$ for $n \leq 5$.

2. One way to define a q-analog of the Eulerian polynomial $A_n(x)$ is

$$A_n(x,q) = \sum_{\sigma \in S_n} x^{d(\sigma)+1} q^{\mathrm{maj}(\sigma)}.$$

(a) Show that with this definition we have

$$\sum_r [r]_q^n x^r = \frac{A_n(x,q)}{(1-x)(1-qx)\cdots(1-q^n x)}$$

(b) Deduce the formula

$$A_n(x,q) = \sum_k [k]_q! S_q(n,k) x^k \prod_{i=k+1}^{n} (1 - xq^i),$$

where $S_q(n,k)$ is the q-analog of a Stirling number defined in Problem Set 1, Problem 11.

3. Show that the coefficients $e_{n,d}(q) = \langle x^{d+1} \rangle A_n(x,q)$ (q-analogs of Eulerian numbers) satisfy $e_{n,d}(q) = q^{nd} e_{n,d}(1/q)$ and $e_{n,n-1-d}(q) = q^{\binom{n}{2}} e_{n,d}(1/q)$.

4. Define the q-*derivative* $(d/dx)_q$ by

$$(d/dx)_q f(x) = \frac{f(x) - f(qx)}{x(1-q)},$$

so that $(d/dx)_q x^n = [n]_q x^{n-1}$, for example. Note that if f is a polynomial, the numerator vanishes both at $x = 0$ and $q = 1$, so it is divisible by the denominaor.

(a) Verify the product rule for q-derivatives

$$(d/dx)_q f(x) g(x) = ((d/dx)_q f(x)) \cdot g(x) + f(qx) \cdot (d/dx)_q g(x).$$

(b) Show that the q-Eulerian polynomials $A_n(x,q)$ defined above satisfy the following q-analog of the recurrence in Problem 1:

$$A_n(x,q) = x(d/dx)_q A_{n-1}(x,q) + q^{\binom{n}{2}} x^n \left((d/dx)_q A_{n-1}(x,q)\right)_{x \mapsto x^{-1},\, q \mapsto q^{-1}}.$$

5. Define the descent set of a word $w \in \mathbb{N}^n$ to be $D(w) = \{i \in [n-1] : w(i) > w(i+1)\}$, just as one does for permutations. Similarly, define $\mathrm{maj}(w) = \sum_{d \in D(w)} d$.

(a) Show that if $w \in [r]^n$ and $w(n) = s$, then $w' = (1\ 2\ \cdots\ r)^{r-s} \circ w$ has $\mathrm{maj}(w') = \mathrm{maj}(w) - (k_{s+1} + \cdots + k_r)$, where k_i is the number of occurences of i in the word w.

(b) Use (a) and the recurrence for q-multinomial coefficients in Problem Set 2, Problem 4 to prove that

$$\sum_{w \in S_n \cdot (1^{k_1}, 2^{k_2}, \ldots, r^{k_r})} t^{\mathrm{maj}(w)} = \sum_{w \in S_n \cdot (1^{k_1}, 2^{k_2}, \ldots, r^{k_r})} t^{\mathrm{inv}(w)},$$

for all $k_1 + \cdots + k_r = n$.

(c) Use (b) to prove that for all $D \subseteq [n-1]$,

$$\sum_{\substack{\pi \in S_n \\ D(\pi^{-1}) = D}} t^{\mathrm{maj}(\pi)} = \sum_{\substack{\pi \in S_n \\ D(\pi^{-1}) = D}} t^{\mathrm{inv}(\pi)},$$

that is, inv and maj are equidistributed on inverse descent classes.

(d) Deduce that

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{maj}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi^{-1})} t^{\mathrm{maj}(\pi)}.$$

Deduce in particular that the left-hand side is symmetric in q and t.

Remark: part (c) implies that $\sum_{\pi \in S_n} x^{d(\pi)+1} q^{\mathrm{maj}(\pi^{-1})} = \sum_{\pi \in S_n} x^{d(\pi)+1} q^{\mathrm{inv}(\pi)}$, which suggests that the common value of these two expressions might be a 'better' q-analog of $A_n(x)$ than the one in the problems above. However, I don't know of nice identities like those above which hold for this alternative q-Eulerian polynomial.

6. A binary tree is an ordered rooted tree in which every non-leaf node has exactly two children. Note that every binary tree has an odd number of vertices. Show that the number of binary trees with $2n+1$ vertices is equal to the Catalan number $C_n = \binom{2n}{n}/(n+1)$ in two ways:

(a) By finding the ordinary generating function counting such trees.

(b) By using Cayley's formula.

7. Prove that the number of ordered rooted trees with $n+1$ vertices and j leaves is equal to

$$\frac{1}{n+1}\binom{n+1}{j}\binom{n-1}{n-j}.$$

8. Prove that the number of ways to subdivide an n-gon into k polygons by introducing $k-1$ diagonals that do not intersect except at their endpoints is equal to the number of ordered rooted trees with $n+k-1$ vertices, $n-1$ leaves, and no vertices with exactly one child. Derive the formula

$$\frac{1}{n+k-1}\binom{n+k-1}{n-1}\binom{n-3}{n-k-2}$$

for this number. In particular, taking $k = n - 2$, deduce that the number of triangulations of an n-gon is the Catalan number C_{n-2}. In this problem the n-gon is regarded as fixed in place, so for example the two triangulations of a square count as different even though they are the same up to symmetry. See Stanley, Exercise 6.19 for 65 more combinatorial interpretations of Catalan numbers.

9. (a) Let X be an $m \times n$ matrix and Y an $n \times m$ matrix. Given a subset $I \subseteq \{1, \ldots, n\}$, let X_I be the submatrix formed by the columns of X with indices $i \in I$ and let Y_I be the submatrix formed similarly by rows of Y. Prove or find a reference for the identity

$$\det XY = \sum_{|I|=m} \det(X_I) \det(Y_I)$$

(b) Let E be the $n \times \binom{n+1}{2}$ matrix constructed as follows: n of the columns are unit vectors e_i, and the remaining $\binom{n}{2}$ columns are differences $e_i - e_j$ for $i < j$. Show that an $n \times n$ square submatrix E_I of E is non-singular if and only if there is a rooted forest F on the vertex set $\{1, \ldots, n\}$ such that the columns of E_I which are unit vectors e_i correspond to the roots i of F and the and the columns which are difference vectors $e_i - e_j$ correspond to the edges $\{i, j\}$. Show in addition that in this case, $\det(E_I) = \pm 1$.

(c) Let $Y = E^t$ and let X be the matrix obtained from E by multiplying each unit vector column e_i by a scalar z_i, and each difference column $e_i - e_j$ by a scalar $-x_{ij}$. Use the formula in part (a) to deduce an alternate proof of the symmetric version of the Matrix-Tree Theorem, that is, its specialization with $x_{ji} = x_{ij}$.

10. The product $G \times H$ of two simple graphs (graphs without loops or multiple edges) is the graph on vertex set $V(G) \times V(H)$ with edges $\{(v, w), (v', w')\}$ for $v = v'$ and $\{w, w'\} \in E(H)$ or $w = w'$ and $\{v, v'\} \in E(G)$. The adjacency matrix A_G of a graph G on n vertices is the $n \times n$ matrix with rows and columns labelled by the vertices, and entries $(A_G)_{v,w} = 1$ if $\{v, w\} \in E(G)$, zero otherwise. Let D_G be the diagonal matrix whose (v, v) entry is the degree of v.

(a) Let $f_G(r)$ be the number of rooted spanning forests of G with r roots, and let $F_G(z) = \sum_r f_G(r) z^r$ be the corresponding generating function. Show that $F_G(z) = \prod_i (z + \alpha_i)$, where the α_i's are the eigenvalues of $D_G - A_G$.

(b) Show that $F_{G\times H}(z) = \prod_{i,j}(z + \alpha_i + \beta_j)$, where $F_G(z) = \prod_i (z + \alpha_i)$ and $F_H(z) = \prod_j (z + \beta_j)$. In particular, the numbers $f_G(r)$ and $f_H(r)$ for all r determine the corresponding numbers $f_{G\times H}(r)$.

(c) Show that if Q_n is the graph formed by the vertices and edges of the n-cube, that is, the product of n copies of the complete graph on 2 vertices, then

$$F_{Q_n}(z) = \prod_{k=0}^{n} (z + 2k)^{\binom{n}{k}}.$$

This generalizes Stanley, Exercise 5.6.10, which follows by taking the coefficient of z.

11. Two species F and G are *equivalent* if they are naturally isomorphic as functors. More precisely, this means that one can give for each finite set S a bijection $\psi_S \colon F(S) \to G(S)$ such that the triple $(F(S), G(S), \psi_S)$ is functorial in S, in the category whose objects are triples consisting of two finite sets and and arrow between them, and whose arrows between two such triples are commutive squares with the given triples as left and right sides.

(a) If L is the species of linear orderings, and P is the species of permutations, show that L and P are not equivalent.

(b) The *Hadamard product* $F * G$ of two species is the species $(F * G)(S) = F(S) \times G(S)$. Show that the species $L * L$ and $P * L$ are equivalent. This can be understood as an explanation of the fact that the inequivalent species L and P have the same exponential generating function.

12. A *distribution* is a function together with a linear ordering on the preimage of each element of the codomain. Recall that the number of surjective distributions from a set n labelled objects to a set of k labelled places is given by

$$\binom{n}{k}(n-1)_{n-k}.$$

We obtained this formula by direct counting. Derive it another way by using exponential generating functions.

13. Find the exponential generating function $D(x) = \sum_n D_n x^n/n!$, where D_n is the number of permutations $\sigma \in S_n$ with no fixed points. Deduce an explicit formula for D_n. [Permutations without fixed points are also called *derangements*. Compare Stanley, Example 2.2.1, where they are counted using the principle of inclusion and exclusion.]

14. An *unordered binary tree* is a rooted tree in which each non-leaf node has two children, but we do not order the children. Find the exponential generating function which counts unordered binary trees on n labelled nodes.

15. An *at most binary tree* is an unordered rooted tree in which each node has at most two children. Find the exponential generating function which counts at most binary trees on n labelled nodes.

16. A *perfect matching* on a set S of $2n$ elements is a partition of S into n blocks of two elements each. Perfect matchings form a species M, with $M(S) = \emptyset$ if $|S|$ is odd.

(a) Find the exponential generating function for the species of perfect matchings.

(b) Deduce algebraically that the number of perfect matchings on a set of $2n$ elements is $n!!$. Here and below the 'double factorial' notation $n!!$ stands for the product $(2n-1)(2n-3)\cdots 3 \cdot 1$ of the first n odd numbers.

(c) Give a direct counting argument for the result in (b).

17. Let $e(2n)$ be the number of permutations σ of a set of $2n$ elements with the property that every cycle of σ has even length.

(a) Find the exponential generating function $\sum_n e(2n)x^{2n}/(2n)!$.

(b) Deduce that $e(2n) = (n!!)^2$.

18. (a) From the preceding problems it follows that the number of permutations of a $2n$ element set S with only even cycles is equal to the number of pairs of perfect matchings on S. Construct a direct bijection between the two (to do this you will probably need to fix the set S to be $[2n]$ and use the numerical values of its elements to make some auxiliary choices).

(b) Show that the species of permutations with even-length cycles and the species of pairs of perfect matchings are not equivalent. This can be understood as explaining the need for auxiliary choices in part (a).

19. (a) Find the exponential generating function $\sum_n r_n x^n/n!$, where r_n is the number of permutations of odd order of an n element set.

(b) Deduce that $r_{2n} = (n!!)^2$ and $r_{2n+1} = (2n+1)(n!!)^2$.

20. A map $f\colon S \to S$ is *idempotent* if $f^2 = f$. Find the exponential generating function for the species of idempotent maps.

21. (a) Show that the exponential generating function for the species of labelled unrooted forests is given by

$$F(x) = \exp \sum_{n=1}^{\infty} n^{n-2} \frac{x^n}{n!}.$$

22. Let $g(n, k)$ denote the number of connected simple graphs (i.e., without loops or multiple edges) on n labelled vertices with k edges. Derive the mixed ordinary/exponential generating function

$$\sum_{n=1}^{\infty} \sum_{k} g(n,k) q^k x^n/n! = \log \sum_{n=0}^{\infty} (1+q)^{\binom{n}{2}} x^n/n!$$

and use it to compute $\sum_k g(n,k)q^k$ for all $n \leq 4$. As a check, count the graphs in question by hand and compare answers.

23. Let $g_+(n)$, $g_-(n)$ denote the number of connected simple graphs on the vertex set $[n]$ with an even or an odd number of edges, respectively. Prove that $g_+(n) - g_-(n) = (-1)^n (n-1)!$.

(b) Use (a) to calculate the number of labelled unrooted forests on $[n]$ for $n \leq 6$.

(c) Modify (a) to get a mixed ordinary/exponential generating function $F(t, x)$ for the species of labelled unrooted forests, in which the coefficient of t^k forest with k components. Use this to refine your answer to (b) to count forests by number of components.

24. (a) Stanley, Exercise 5.5. A proper n-coloring of a graph $G = (V, E)$ is a function $c : V \to [n]$ such that $c(v) \neq c(w)$ whenever v, w are the endpoints of an edge $e \in E$. A graph G is *bipartite* if there exists a proper 2-coloring of G.

(b) Let B be the species $B(S) = \{$bipartite graphs with vertex set $S\}$ and let G_2 be the species $G_2(S) = \{$graphs G with vertex set S, together with a proper 2-coloring of $G\}$. Part (a) implies that G_2 and B^2 have the same exponential generating function, and indeed the same mixed generating function weighted by number of edges. Are the species B^2 and G_2 equivalent?

(c) Derive the mixed ordinary/exponential generating function for the species of *connected* bipartite graphs, weighted by $q^{\text{number of edges}}$. Use it to calculate the number of connected labelled bipartite graphs with k edges on n vertices for $n \leq 5$ and all k.

25. (a) From Cayley's tree generating function derive the identity

$$\sum_F \prod_{i=1}^n x_i^{c_F(i)} = \binom{n-1}{k-1}(x_1 + \cdots x_n)^{n-k},$$

where the sum is over rooted forests F with k components on vertices $\{1, \ldots, n\}$, and $c_F(i)$ denotes the number of children of vertex i in F.

(b) Deduce the identity

$$\sum_F \prod_{i=1}^n h_{c_F(i)} = \binom{n-1}{k-1}\langle\frac{x^{n-k}}{(n-k)!}\rangle H(x)^n,$$

where $H(x) = \sum_m h_m x^m/m!$ is a generic formal power series written in exponential form and $\langle\cdot\rangle$ denotes taking a coefficient.

(c) Let H be the 'generic species' with exponential generating function $H(x)$, that is, the trivial species, but enumerated by assigning weight h_n to the one structure on any set with n elements. Let $F(x)$ be the solution of the formal functional equation

$$F(x) = x\,H(F(x))$$

(that is, assuming h_0 invertible, the functional composition inverse of $x/H(x)$). Using (b) and a species interpretation of $F(x)$, obtain the generalized Lagrange inversion formula

$$\langle\frac{x^n}{n!}\rangle F(x)^k/k! = \binom{n-1}{k-1}\langle\frac{x^{n-k}}{(n-k)!}\rangle H(x)^n,$$

or equivalently,

$$\langle x^n\rangle F(x)^k = \frac{k}{n}\langle x^{n-k}\rangle H(x)^n.$$

26. A *leaf-labelled* tree T is a rooted tree whose leaves are the elements of a given set S, and whose other nodes are unlabelled. Let us require each non-leaf node of T to have at least two children; then there are finitely many leaf-labelled trees on a given finite leaf set S.

Introduce indeterminates $a_2, a_3, \ldots$ and assign each leaf-labelled tree the weight $w(T) = \prod_i a_i^{r_i}$, where r_i is the number of non-leaf nodes in T with i children. Then show that the mixed ordinary/exponential generating function enumerating leaf-labelled trees with these weights is the functional composition inverse of $x - A(x)$, where $A(x) = \sum_{n\geq 2} a_n x^n/n!$.