Math 249 Problem Set 2

Problems from Stanley (Vol. 1, second edition).

1.5, 1.22(b), 1.48(b), 1.51, 1.76, 1.102

Additional problems.

1. By expanding the right hand side in partial fractions, show directly that the generating function for Stirling numbers

$$\sum_n S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-k\,x)}$$

is equivalent to the explicit formula

$$S(n,k) = \frac{1}{k!}\sum_j (-1)^{k-j}\binom{k}{j} j^n.$$

2. Construct a formula for the generating function $\sum_n a_n x^n$, where a_n is the number of combinations of states and the District of Columbia having a combined number n of electoral votes (you can find the electoral vote distribution on the Wikipedia "Electoral College" page). Using a computer algebra system, evaluate the generating function explicitly and answer the following question: in a presidential election with two candidates, how many combinations lead to a tie in the electoral college, and how many to a win for either candidate?

To get it precisely right, you should take into account that Maine and Nebraska have district systems allowing them to split their electoral votes (Maine's 4 votes cannot split 2 and 2, however).

A news reporter contacted the MIT math department with this question when I was a grad student there. The department chair passed the problem along to us combinatorics grad students to solve.

3. Let l divide n. Show that the primitive l-th roots of unity are roots of the polynomial $\binom{n}{k}_q$ in q if and only if l does not divide k.

4. Prove that the q-multinomial coefficients satisfy the recurrence

$$\binom{n}{k_1,k_2,\ldots,k_r}_q = \binom{n-1}{k_1-1,k_2,\ldots,k_r}_q + q^{k_1}\binom{n-1}{k_1,k_2-1,\ldots,k_r}_q + \cdots + q^{k_1+\cdots+k_{r-1}}\binom{n-1}{k_1,k_2,\ldots,k_r-1}_q.$$

5. Prove the following q-analog of the convolution formula for binomial coefficients.

$$\binom{m+n}{k}_q = \sum_{i+j=k} q^{(m-i)j}\binom{m}{i}_q\binom{n}{j}_q$$

6. (a) Show that for each k there is a unique polynomial $Q_k(x)$ of degree k, with coefficients in the field of rational functions $\mathbb{Q}(q)$, such that $Q_k(q^n) = \binom{n}{k}_q$ for all n.

(b) Prove that a polynomial $f \in \mathbb{Q}(q)[x]$ has the property that $f(q^n) \in \mathbb{Z}[q, q^{-1}]$ for all n if and only if the coefficients of f with respect to the basis $\{Q_k : k \in \mathbb{N}\}$ belong to $\mathbb{Z}[q, q^{-1}]$. Hint: evaluate f at roots of the polynomials $Q_k(x)$.

7. Prove that the number of partitions of n in which each part j is repeated less than j times is equal to the number of partitions of n in which no part is a square.

8. From the q-binomial theorem

$$\prod_{i=0}^{m-1}(1 + xq^i) = \sum_{j=0}^{m} \binom{m}{j}_q q^{\binom{j}{2}} x^j,$$

deduce

$$\prod_{i=1}^{s}(1 + x^{-1}q^i) \prod_{i=0}^{t-1}(1 + xq^i) = \sum_{j=-s}^{t} \binom{s+t}{s+j}_q q^{\binom{j}{2}} x^j.$$

By letting s and t go to infinity, prove *Jacobi's triple product identity:*

$$\sum_{j\in\mathbb{Z}}(-1)^j a^{\binom{j}{2}} x^j = \prod_{i\geq 0}(1 - xa^i)(1 - x^{-1}a^{i+1})(1 - a^{i+1})$$

9. Deduce Euler's pentagonal number theorem by letting a and x go to suitable powers of q in Jacobi's triple product identity.

10. The following two identities are due to Gauss:

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} = \prod_{i\geq 1}\frac{1 - q^i}{1 + q^i};$$

$$\sum_{n\geq 0} q^{\binom{n+1}{2}} = \prod_{i\geq 1}\frac{1 - q^{2i}}{1 - q^{2i-1}}.$$

(a) Interpret them combinatorially as partition identities.

(b) Prove them, either combinatorially (not so easy) or using Jacobi's triple product identity.

11. For a subset $S \subseteq [n]$, let $\|S\|$ denote the sum of the elements of S. Show that

$$\sum_{\substack{S\subseteq[n]\\|S|=k}} q^{\|S\|} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

12. Prove Euler's pentagonal number theorem in the form

$$1 = \left(\sum_{j\in\mathbb{Z}}(-1)^j q^{3\binom{j}{2}+j}\right) \cdot \prod_{i\geq 1}\frac{1}{1 - q^i}$$

by constructing a sign-reversing involution on the set of pairs $(j; \lambda)$, where j is an integer and λ is a partition, weighted by $(-1)^j q^{|\lambda|+3\binom{j}{2}+j}$, excluding the pair $(0; \emptyset)$. Your involution $S : (j; \lambda) \mapsto (\hat{j}; \hat{\lambda})$ should have the property that $\hat{j} = j \pm 1$, and $\hat{\lambda}$ is obtained from λ by removing the longest row and adding a column, or vice versa.