Math 249 Problem Set 1

Problems from Stanley. Notation X [*Y*] means Exercise X in the second edition, corresponding to Y in the first (see Appendix "First Edition Numbering" in the second edition).

1.8 [*1.4*], 1.19(a) [*1.8(a)*], 1.32 [*1.12*], 1.46(a) [*1.17(a)*]

In 1.32 [*1.12*], what if the subsets $T_i \subseteq S$ in 1.12 are required to be non-empty, proper and distinct?

Additional problems.

1. Recall that a sequence of formal power series $F_k(x) = \sum_n a_n^{(k)} x^n$ converges to $G(x) = \sum_n b_n x^n$ if for each n the sequence $(a_n^{(k)})$ converges to b_n in the discrete topology, i.e., $a_n^{(k)} = b_n$ for all sufficiently large k.

(a) Show that the partial sums $a_0 + a_1 x + \cdots + a_n x^n$ converge to $\sum_n a_n x^n$.

(b) Show that a sum $\sum_{k=1}^{\infty} F_k(x)$ converges if and only if the terms $F_k(x)$ converge to 0. Show that the property of convergence and the value of the limit do not depend on the order of the terms.

(c) Show that if the factors $F_k(x)$ converge to 1, then the product $\prod_{k=1}^{\infty} F_k(x)$ converges, its value does not depend on the order of the factors, and that it is non-zero if the coefficient ring R is an integral domain. Show that if infinitely many of the factors F_k have zero constant term, then the product converges to zero. Show that if R is an integral domain, then the product converges only in these two cases.

(d) Show that a sum or product of limits of convergent sequences is the limit of the sums or products term by term. Show that this also holds for convergent infinite sums and products.

2. If $F(x)$ and $G(x)$ are formal power series and $F(0) = 0$, i.e., $F(x)$ has zero constant term, their formal composition is defined by

$$(G \circ F)(x) = \sum_{k=0}^{\infty} g_k F(x)^k,$$

where $G(x) = \sum_{k=0}^{\infty} g_k x^k$. Note that the sum converges by part (b) of the preceding problem. We also write $G(F(x))$ for $(G \circ F)(x)$.

(a) Show that composition is associative, i.e., if $F(0) = 0$ and $G(0) = 0$, then $(H \circ G) \circ F = H \circ (G \circ F)$.

(b) Show that composition with a fixed F, considered as a function of G, is a ring homomorphism.

(c) Show that $F(x) \in R[[x]]$ such that $F(0) = 0$ has a formal compositional inverse if and only if the coefficient $\langle x \rangle F(x)$ has a multiplicative inverse in R.

3. Show that $F(x) \in R[[x]]$ has a multiplicative inverse if and only if its constant term $F(0)$ has a multiplicative inverse in R.

4. Defining the formal derivative $\frac{d}{dx}F(x)$ of a formal power series term by term in the obvious way, show that the usual sum and product rules and the chain rule hold. Show that Taylor's formula holds if the coefficient ring contains $\mathbb{Q}$.

5. Express each of the following as a binomial coefficient: (a) the number of monomials of degree exactly d in a (commutative) polynomial ring in n variables; (b) the number of monomials of degree $\leq d$.

6. Find the number of monotone maps $f\colon \{1,\ldots,k\} \to \{1,\ldots,n\}$, where *monotone* means $f(i) \leq f(j)$ for $i \leq j$.

7. (a) Give two proofs of the binomial coefficient identity, called the *convolution formula*,

$$\sum_j \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k}.$$

One proof should use generating functions, the other should be a direct combinatorial proof.

(b) Discover and prove in the same two ways an analogous identity for multiset coefficients $\left\langle {n \atop k} \right\rangle$.

8. Find a simple expression for the ordinary generating function in two variables

$$\sum_{n,k\geq 0} \binom{n}{k} x^n y^k,$$

and use it to deduce the identity

$$\sum_r \binom{r}{k} x^r = \frac{x^k}{(1-x)^{k+1}}.$$

9. Show that $\left\langle {n \atop k} \right\rangle = \sum_{j=0}^k \left\langle {n-1 \atop j} \right\rangle$. Using this, evaluate $\sum_{j=0}^k \binom{n+j}{j}$ as a single binomial coefficient.

10. Regarding $\binom{x}{k}$ as a polynomial of degree k in the variable x, prove that a polynomial $f \in \mathbb{Q}[x]$ has the property that $f(n)$ is an integer for all integers n if and only if the coefficients of f with respect to the basis $\{\binom{x}{k} : k \in \mathbb{N}\}$ are integers.

Hint: for "only if," express the coefficients a_k such that $f(x) = \sum_k a_k \binom{x}{k}$ in terms of the iterated differences $(\Delta^m f)(0)$, where $\Delta f(x) = f(x+1) - f(x)$.

11. Show that the Stirling numbers of the second kind $S(n,k)$ have a q-analog $S_q(n,k)$ characterized by the following properties, where $[k]_q = (1-q^k)/(1-q) = 1 + q + \cdots + q^{k-1}$ is the q analog of k.

(a) They satisfy the recurrence

$$S_q(n,k) = [k]_q S_q(n-1,k) + q^{k-1} S_q(n-1,k-1),$$

with inital condition $S_q(0,n) = S_q(n,0) = \delta_{0,n}$.

(b) They satisfy the following q-analog of the classical formula $x^n = \sum_k S(n,k)(x)_k$:

$$[r]_q^n = \sum_k S_q(n,k)[r]_q[r-1]_q \cdots [r-k+1]_q$$

(c) For each k, they are given by the ordinary generating function

$$\sum_n S_q(n,k)x^n = \frac{q^{\binom{k}{2}} x^k}{(1-x)(1-[2]_q x)\cdots(1-[k]_q x)}.$$

(d) Given a partition $\pi = \{B_1, \ldots, B_k\}$ of $[n]$, with the blocks numbered so that $\min(B_i) < \min(B_j)$ for $i < j$, define $\nu(\pi) = \sum_i (i-1)|B_i|$. Then $S_q(n,k) = \sum_\pi q^{\nu(\pi)}$, where the sum is over partitions of $[n]$ into k blocks.