Math 249 Spring 2012 Homework Problems

Lecture 1

1. Stanley, Ch 1, Ex. 1.2(a)
2. Stanley Ex. 1.7
3. Stanley Ex. 1.9
4. Stanley Ex. 1.12
5. Stanley Ex. 1.13
6. Stanley Ex. 1.29

7. Find a recurrence similar to Pascal's triangle for the signless Stirling numbers $c(n,k)$ of the first kind (*i.e.*, the number of permutations $\pi \in S_n$ with k cycles), and use it to compute a table of $c(n,k)$ for $k, n \leq 5$.

Lecture 2

1. Stanley Ex. 1.8(a,b)
2. Stanley Ex. 1.16
3. Stanley Ex. 1.19
4. Stanley Ex. 1.24
5. Find a simple expression for the ordinary generating function in two variables

$$\sum_{n,k\geq 0} \binom{n}{k} x^n y^k,$$

and use it to deduce the identity

$$\sum_r \binom{r}{k} x^r = \frac{x^k}{(1-x)^{k+1}}.$$

6. Prove the ordinary generating function identity for signless Stirling numbers of the first kind

$$\sum_k c(n,k) y^k = y(y+1)\cdots(y+n-1).$$

7. A *perfect matching* on a set S of $2n$ elements is a partition of S into n blocks of 2 elements each. Taking $S = [2n] = \{1, 2, \cdots, 2n\}$, and thinking of the blocks in a matching as the edges of a graph, call edges of the form $\{i, i+1\}$ *short*, and all other edges *long*.

(a) Show that the number of perfect matchings on a $2n$-element set is

$$(2n-1)(2n-3)\cdots 3\cdot 1.$$

(b) Let $M_n(x)$ be the ordinary generating function that counts perfect matchings on $[2n]$ with weight x^s, where s is the number of short edges, so for instance $M_2(x) = 1 + x + x^2$. Prove the recurrence

$$M_n(x) = (x + 2n - 2)M_{n-1}(x) + (1 - x)\frac{d}{dx}M_{n-1}(x).$$

8. Let $m_d(q)$ be the number of irreducible monic polynomials $f(x)$ of degree d, over the finite field $\mathbb{F}(q)$ with q elements. Note that the number of *all* monic polynomials of degree d (irreducible or not) is just q^d.

(a) Use unique factorization of polynomials to prove the generating function identity

$$\prod_{d \geq 1} \frac{1}{(1 - x^d)^{m_d(q)}} = \frac{1}{1 - qx}.$$

(b) By taking logarithms on both sides, derive the identity

$$\sum_{d|n} d m_d(q) = q^n$$

for all n, the sum ranging over the divisors of n. Equivalently,

$$m_d(q) = \frac{1}{d}\sum_{m|d} \mu(d/m)q^m,$$

where $\mu(n)$ is the Möbius function from number theory, *i.e.*, $\mu(n) = (-1)^k$ if n is a product of k distinct primes, and $\mu(n) = 0$ if n is divisible by a square.

(c) Use (b) to prove that the product of all monic irreducible polynomials of degree dividing n is equal to $x^{q^n} - x$. [Hint: every element of $\mathbb{F}(q^n)$ is a root of $x^{q^n} - x$.]

(d) A *necklace* is an equivalence class of words up to rotation. A necklace of length n is *primitive* if the corresponding rotation class consists of n distinct words, *i.e.*, it is not periodic with period d a proper divisor of n. (Example: 1122 is primitive; 1212 is not.) Note that every word of length n consists of n/d repetitions of a primitive necklace of length d dividing n. (Example: 1212 and 2121 both repeat the primitive necklace $12 = 21$.) Let $p_d(q)$ be the number of primitive necklaces of length d on an alphabet of q symbols. Prove that

$$\sum_{d|n} d p_d(q) = q^n,$$

and hence

$$m_d(q) = p_d(q)$$

when q is a power of a prime.

(e) (For those familiar with Lie algebras.) Let $L_q = L(x_1, \ldots, x_q)$ be the free Lie algebra with q generators. L_q is graded if we consider each generator to be homogeneous of degree 1.

Let $l_d(q)$ be the dimension of the homogeneous part of degree d in L_q. In other words, $l_d(q)$ is the number of linearly independent expressions that can be formed by bracketing together d of the generators x_i. The universal enveloping algebra of L_q is the free tensor algebra $T(x_1, \ldots, x_q)$. Using this and the Poincaré-Birkhoff-Witt theorem, derive the identity

$$\prod_{d \geq 1} \frac{1}{(1-x^d)^{l_d(q)}} = \frac{1}{1-qx}.$$

Deduce that $l_d(q) = p_d(q)$.

Remark: This enumerative result suggests that L_q should have a basis whose elements in degree d are indexed in some natural way by primitive necklaces of length d. Such a basis has been constructed by R. Lyndon.

Lecture 3

1. Stanley 1.25

2. Stanley 1.26

3. Stanley 1.32 (a nice example of a combinatorially meaningful application of formal power series with radius of convergence zero).

4. Stanley 1.39

5. Stanley 1.40, but show that the a_n and f_n can be expresed as polynomials in each other over the integers, so that the identity holds between formal power series in x over any commutative ring R when the a_n and f_n are elements of R.

6. Let $\langle x^n \rangle F(x)$ denote the coefficient of x^n in a formal power series $F(x)$. A sequence $(F_k(x))$ in $R[[x]]$ is said to *converge* to $G(x)$ if, for each n, the sequence $(\langle x^n \rangle F_k(x))$ converges to $\langle x^n \rangle G(x)$ in the discrete topology on R; that is, if $\langle x^n \rangle F_k(x) = \langle x^n \rangle G(x)$ for all sufficiently large k. The definition adapts in an obvious way to formal power series in several variables.

(a) Show that the partial sums $f_0 + f_1 x + \cdots + f_k x^k$ of $F(x)$ converge to $F(x)$.

(b) Show that a sum $\sum_{k=1}^{\infty} F_k(x)$ converges if and only if $(F_k(x))$ converges to 0. Show that rearranging the terms does not change the property of convergence, nor the value of the limit.

(c) Show that a product $\prod_{k=1}^{\infty} F_k(x)$ converges if $(F_k(x))$ converges to 1; if R is an integral domain, show that this is if and only if. Again show that the limit and the property of convergence are independent of rearranging the terms.

(d) Show that a sum or product of limits of convergent sequences is the limit of the sums or products term by term. Show that this also holds for convergent infinite sums and products.

7. If $F(x)$ and $G(x)$ are formal power series, and $F(0) = 0$ (where $F(0)$ is defined to be the constant term of $F(x)$), their formal composition is defined by

$$(G \circ F)(x) = \sum_{k=0}^{\infty} g_k F(x)^k,$$

where $G(x) = \sum_{k=0}^{\infty} g_k x^k$. Note that the sum converges by part (b) of the preceding problem. We also write $G(F(x))$ for $(G \circ F)(x)$.

(a) Show that composition is associative, *i.e.*, if $F(0) = 0$ and $G(0) = 0$, then $(H \circ G) \circ F = H \circ (G \circ F)$.

(b) Show that composition with a fixed F, considered as a function of G, is a ring homomorphism.

(c) Show that $F(x) \in R[[x]]$ such that $F(0) = 0$ has a formal compositional inverse if and only if the coefficient $\langle x \rangle F(x)$ has a multiplicative inverse in R.

8. Show that $F(x) \in R[[x]]$ has a multiplicative inverse if and only if its constant term $F(0)$ has a multiplicative inverse in R.

9. Assume we are working in a formal power series ring $R[[x]]$ over a coefficient ring R containing $\mathbb{Q}$. Define $\exp(x)$ and $\log(1+x)$ by their usual Taylor series.

(a) Given $F(x), G(x) \in R[[x]]$ such that $F(0) = 1$, show that there is a well-defined formal power series

$$F(x)^{G(x)} \underset{\text{def}}{=} \exp(G(x) \log F(x)).$$

(b) Show that the above definition satisfies the usual laws of exponents, namely, $F(x)^{G(x)+H(x)} = F(x)^{G(x)} F(x)^{H(x)}$, $F(x)^{G(x)H(x)} = (F(x)^{G(x)})^{H(x)}$, $F(x)^0 = 1$, $F(x)^1 = F(x)$.

10. Defining the derivative $\frac{d}{dx} F(x)$ of a formal power series in the obvious formal way, show that the usual sum and product rules and the chain rule hold. If the coefficient ring contains $\mathbb{Q}$, show that Taylor's formula holds.

11. (a) Prove that the number of partitions of n with no parts divisible by d is equal to the number of partitions of n with no part repeated d or more times, for all n and d.

(b) Prove that the number of partitions of n in which each part j is repeated less than j times is equal to the number of partitions of n in which no part is a square.

12. Let $p_+(n)$ be the the number of partitions of n with an even number of parts and $p_-(n)$ the number with an odd number of parts. Let $p_{DO}(n)$ be the number of partitions of n with distinct odd parts, and let $k(n)$ be the number of partitions of n which are conjugate to themselves. Prove that

$$k(n) = p_{DO}(n) = (-1)^n (p_+(n) - p_-(n))$$

13. The *Durfee square* of a partition λ is the largest $k \times k$ square that fits inside its Young diagram. Use Durfee squares to prove the identities in Stanley, Ch. 1, Ex. 23(b,d).

Lecture 4

1. The group of *signed permutations* B_n is a Coxeter group which acts on $\mathbb{R}^n$ with Coxeter generators the transpositions $\sigma_i = (i\ i+1) \in S_n$ and the sign change $\tau(x_1, x_2, \ldots, x_n) = (-x_1, x_2 \ldots, x_n)$. Its elements may be represented by their actions on the vector $(1, 2, \ldots, n)$ as words w in the signed alphabet $\{\pm 1, \ldots, \pm n\}$ such that the absolute values $|w_i|$ form a

permutation. Define $\operatorname{inv}(w) = |\{i < j : w(i) > w(j)\}| + |\{i < j : w(i) + w(j) < 0\}| + |\{i : w(i) < 0\}|$.

(a) Show that $\operatorname{inv}(w)$ is the equal to the minimum length of an expression for w as a product of the Coxeter generators.

(b) Prove the identity

$$\sum_{w \in B_n} q^{\operatorname{inv}(w)} = (2n)_q (2n-2)_q \cdots (2)_q.$$

2. Prove that the q-multinomial coefficients satisfy the recurrence

$$\binom{n}{k_1, k_2, \ldots, k_r}_q = \binom{n-1}{k_1 - 1, k_2, \ldots, k_r}_q + q^{k_1} \binom{n-1}{k_1, k_2 - 1, \ldots, k_r}_q + \cdots + q^{k_1 + \cdots + k_{r-1}} \binom{n-1}{k_1, k_2, \ldots, k_r - 1}_q.$$

3. Prove the following q-analog of the convolution formula for binomial coefficients.

$$\binom{m+n}{k}_q = \sum_{i+j=k} q^{(m-i)j} \binom{m}{i}_q \binom{n}{j}_q$$

4. Let l divide n. Show that the primitive l-th roots of unity are roots of the polynomial $\binom{n}{k}_q$ in q if and only if l does not divide k.

5. Regarding $\binom{x}{k}$ as a polynomial of degree k in the variable x, prove that a polynomial $f \in \mathbb{Q}[x]$ has the property that $f(n)$ is an integer for all integers n if and only if the coefficients of f with respect to the basis $\{\binom{x}{k} : k \in \mathbb{N}\}$ are integers.

Hint: one direction is easy. For the other, use the fact that $0, 1, \ldots, k-1$ are roots of $\binom{x}{k}$, and $\binom{k}{k} = 1$.

6. (a) Show that for each k there is a unique polynomial $Q_k(x)$ of degree k, with coefficients in the field of rational functions $\mathbb{Q}(q)$, such that $Q_k(q^n) = \binom{n}{k}_q$ for all n.

(b) Prove that a polynomial $f \in \mathbb{Q}(q)[x]$ has the property that $f(q^n) \in \mathbb{Z}[q, q^{-1}]$ for all n if and only if the coefficients of f with respect to the basis $\{Q_k : k \in \mathbb{N}\}$ belong to $\mathbb{Z}[q, q^{-1}]$.

7. Prove *Cauchy's identity:*

$$\prod_{i \geq 0} \frac{1 - axq^i}{1 - xq^i} = \sum_{n \geq 0} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} x^n,$$

by showing that it reduces to the q-binomial theorem upon setting $a = q^m$ for an integer m (note that you get either form of the q-binomial theorem by taking m positive or negative).

8. From the q-binomial theorem

$$\prod_{i=0}^{m-1}(1+xq^i)=\sum_{j=0}^{m}\binom{m}{j}_q q^{\binom{j}{2}}x^j,$$

deduce

$$\prod_{i=1}^{s}(1+x^{-1}q^i)\prod_{i=0}^{t-1}(1+xq^i)=\sum_{j=-s}^{t}\binom{s+t}{s+j}_q q^{\binom{j}{2}}x^j.$$

By letting s and t go to infinity, prove *Jacobi's triple product identity:*

$$\sum_{j\in\mathbb{Z}}(-1)^j a^{\binom{j}{2}}x^j=\prod_{i\geq 0}(1-xa^i)(1-x^{-1}a^{i+1})(1-a^{i+1})$$

9. The following two identities are due to Gauss:

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}=\prod_{i\geq 1}\frac{1-q^i}{1+q^i};$$

$$\sum_{n\geq 0}q^{\binom{n+1}{2}}=\prod_{i\geq 1}\frac{1-q^{2i}}{1-q^{2i-1}}.$$

(a) Interpret them combinatorially as partition identities.

(b) Prove them, either combinatorially (not so easy) or using Jacobi's triple product identity.

10. Let $\mathbb{Q}(q)\langle x,y\rangle$ be the algebra of polynomials in non-commuting variables x,y, over the field of rational functions $\mathbb{Q}(q)$, and let $Q_q[x,y]=\mathbb{Q}(q)\langle x,y\rangle/J$, where J is the two-sided ideal generated by $yx-qxy$. Thus $Q_q[x,y]$ is the 'quantum polynomial ring' whose generators satisfy the q-commutation relation $yx=qxy$. Prove the 'quantum q-binomial theorem' that

$$(x+y)^n=\sum_k\binom{n}{k}_q x^k y^{n-k}$$

holds as an identity in $Q_q[x,y]$.

Lecture 5

1. Stanley, 1.33. The definition of ordered set partition should be formulated less ambiguously: a set partition together with a linear ordering on its set of blocks.

2. Prove that the Eulerian polynomials $A_n(x)=\sum_{\sigma\in S_n}x^{d(\sigma)+1}$ satisfy the recurrence

$$A_n(x)=nxA_{n-1}(x)+x(1-x)A'_{n-1}(x).$$

3. Prove that the Eulerian polynomials $A_n(x)$ satisfy the more symmetrical recurrence

$$A_n(x)=xA'_{n-1}(x)+x^nA'_{n-1}(x^{-1})$$

for $n > 1$, and use it to compute $A_n(x)$ for $n \le 5$.

Lecture 6

1. Show that the Stirling numbers of the second kind $S(n,k)$ have a q-analog $S_q(n,k)$ characterized by the following properties:

(a) They satisfy the recurrence

$$S_q(n,k) = (k)_q S_q(n-1,k) + q^{k-1} S_q(n-1,k-1),$$

with inital conditions $S_q(0,k) = \delta_{0,k}$ and $S_q(n,0) = \delta_{n,0}$.

(b) They satisfy the following q-analog of the classical formula $x^n = \sum_k S(n,k)(x)_k$:

$$((r)_q)^n = \sum_k S_q(n,k)(r)_q(r-1)_q \cdots (r-k+1)_q$$

(c) They are given by the ordinary generating function (for each k)

$$\sum_n S_q(n,k)x^n = \frac{q^{\binom{k}{2}} x^k}{(1-x)(1-(2)_q x)\cdots(1-(k)_q x)}.$$

(d) Given a partition $\pi = \{B_1, \ldots, B_k\}$ of $[n]$, with the blocks numbered so that $\min(B_i) < \min(B_j)$ for $i < j$, define $\nu(\pi) = \sum_i (i-1)|B_i|$. Then $S_q(n,k) = \sum_\pi q^{\nu(\pi)}$, where the sum is over partitions of $[n]$ into k blocks.

2. One way to define a t-analog of the Eulerian polynomials $A_n(x)$ is

$$A_n(x,t) = \sum_{\sigma \in S_n} x^{d(\sigma)+1} t^{\mathrm{maj}(\sigma)}.$$

(a) Show (generalizing what we did in the lecture) that with this definition we have

$$\sum_r (r)_t^n x^r = \frac{A_n(x;t)}{(1-x)(1-tx)\cdots(1-t^n x)}$$

(b) Deduce the formula

$$A_n(x,t) = \sum_k (k)_t! S_t(n,k) x^k \prod_{i=k+1}^{n} (1-xt^i).$$

3. Show that the coefficients $e_{n,d}(t) = \langle x^{d+1} \rangle A_n(x,t)$ satisfy $e_{n,d}(t) = t^{nd} e_{n,d}(1/t)$ and $e_{n,n-1-d}(t) = t^{\binom{n}{2}} e_{n,d}(1/t)$.

4. Define the descent set of a word $w \in \mathbb{N}^n$ to be $D(w) = \{i \in [n-1] : w(i) > w(i+1)\}$, just as one does for permutations. Similarly, define $\mathrm{maj}(w) = \sum_{d \in D(w)} d$.

(a) Show that if $w \in [r]^n$ and $w(n) = s$, then $w' = (1\ 2\ \cdots\ r)^{r-s} \circ w$ has $\operatorname{maj}(w') = \operatorname{maj}(w) - (k_{s+1} + \cdots + k_r)$, where k_i is the number of occurences of i in the word w.

(b) Use (a) and the recurrence in Lecture 4, problem 2 to prove that

$$\sum_{w \in S_n \cdot (1^{k_1}, 2^{k_2}, \ldots, r^{k_r})} t^{\operatorname{maj}(w)} = \sum_{w \in S_n \cdot (1^{k_1}, 2^{k_2}, \ldots, r^{k_r})} t^{\operatorname{inv}(w)},$$

for all $k_1 + \cdots + k_r = n$.

(c) Use (b) to prove that for all $D \subseteq [n-1]$,

$$\sum_{\substack{\pi \in S_n \\ D(\pi^{-1}) = D}} t^{\operatorname{maj}(\pi)} = \sum_{\substack{\pi \in S_n \\ D(\pi^{-1}) = D}} t^{\operatorname{inv}(\pi)},$$

that is, inv and maj are equidistributed on inverse descent classes.

(d) Deduce that

$$\sum_{\pi \in S_n} q^{\operatorname{inv}(\pi)} t^{\operatorname{maj}(\pi)} = \sum_{\pi \in S_n} q^{\operatorname{maj}(\pi^{-1})} t^{\operatorname{maj}(\pi)}.$$

Deduce in particular that the left-hand side is symmetric in q and t.

Remark: part (c) implies that $\sum_{\pi \in S_n} x^{d(\pi)+1} q^{\operatorname{maj}(\pi^{-1})} = \sum_{\pi \in S_n} x^{d(\pi)+1} q^{\operatorname{inv}(\pi)}$, which suggests that the common value of these two expressions might be a 'better' q-analog of $A_n(x)$ than the one in Problem 2. However, I don't know of nice identities like those in Problems 2 and 3 which hold for this alternative q-Eulerian polynomial.

Lecture 7

1. Two species F and G are *equivalent* if they are naturally isomorphic as functors. More precisely, this means that one can give for each finite set S a bijection $\psi_S \colon F(S) \to G(S)$ such that the triple $(F(S), G(S), \psi_S)$ is functorial in S, in the category whose objects are triples consisting of two finite sets and and arrow between them, and whose arrows between two such triples are commutive squares with the given triples as left and right sides.

(a) If L is the species of linear orderings, and P is the species of permutations, show that L and P are not equivalent.

(b) The *Hadamard product* $F * G$ of two species is the species $(F * G)(S) = F(S) \times G(S)$. Show that the species $L * L$ and $P * L$ are equivalent. This can be understood as an explanation of the fact that the inequivalent species L and P have the same exponential generating function.

2. Prove that the operations of sum and product of species are commutative and associative with two-sided identities, up to equivalence of species.

3. A *distribution* is a function together with a linear ordering on the preimage of each element of the codomain. Recall that the number of surjective distributions from a set n labelled objects to a set of k labelled places is given by

$$\binom{n}{k}(n-1)_{n-k}.$$

We obtained this formula in our discussion of the 12-fold way table by direct counting. Derive it another way by using exponential generating functions.

4. Find the exponential generating function $D(x) = \sum_n D_n x^n/n!$, where D_n is the number of permutations $\sigma \in S_n$ with no fixed points. Deduce an explicit formula for D_n. [Permutations without fixed points are also called *derangements*. Compare Stanley, Example 2.2.1, where they are counted using the principle of inclusion and exclusion.]

Lecture 8

1. An *unordered binary tree* is a rooted tree in which each non-leaf node has two children, but we do not order the children. Find the exponential generating function which counts unordered binary trees on n labelled nodes.

2. An *at most binary tree* is an unordered rooted tree in which each node has at most two children. Find the exponential generating function which counts at most binary trees on n labelled nodes.

3. A *perfect matching* on a set S of $2n$ elements is a partition of S into n blocks of two elements each. Perfect matchings form a species M, with $M(S) = \emptyset$ if $|S|$ is odd.

(a) Find the exponential generating function for the species of perfect matchings.

(b) Deduce algebraically that the number of perfect matchings on a set of $2n$ elements is $n!!$. Here and below the 'double factorial' notation $n!!$ stands for the product $(2n-1)(2n-3)\cdots 3\cdot 1$ of the first n odd numbers.

(c) Give a direct counting argument for the result in (b).

4. An *involution* of S is a permutation τ of S such that $\tau^2 = 1$.

(a) Find the exponential generating function $\sum_n t_n x^n/n!$, where t_n is the number of involutions of an n element set.

(b) Show that $t_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k!!$.

5. Let $e(2n)$ be the number of permutations σ of a set of $2n$ elements with the property that every cycle of σ has even length.

(a) Find the exponential generating function $\sum_n e(2n)x^{2n}/(2n)!$.

(b) Deduce that $e(2n) = (n!!)^2$.

6. (a) From the preceding problems it follows that the number of permtutations of a $2n$ element set S with only even cycles is equal to the number of pairs of perfect matchings on S. Construct a direct bijection between the two (to do this you will probably need to fix the set S to be $[2n]$ and use the numerical values of its elements to make some auxiliary choices).

(b) Show that the species of permutations with even-length cycles and the species of pairs of perfect matchings are not equivalent. This can be understood as explaining the need for auxiliary choices in part (a).

7. (a) Find the exponential generating function $\sum_n r_n x^n/n!$, where r_n is the number of permutations of odd order of an n element set.

(b) Deduce that $r_{2n} = (n!!)^2$ and $r_{2n+1} = (2n+1)(n!!)^2$.

8. Let $g(n,k)$ denote the number of connected simple graphs (*i.e.*, without loops or multiple edges) on n labelled vertices with k edges. Derive the mixed ordinary/exponential generating function

$$\sum_{n=1}^{\infty}\sum_{k} g(n,k)q^k x^n/n! = \log \sum_{n=0}^{\infty}(1+q)^{\binom{n}{2}} x^n/n!$$

and use it to compute $\sum_k g(n,k)q^k$ for all $n \leq 4$. As a check, count the graphs in question by hand and compare answers.

9. Let $g_+(n)$, $g_-(n)$ denote the number of connected simple graphs on the vertex set $[n]$ with an even or an odd number of edges, respectively. Prove that $g_+(n) - g_-(n) = (-1)^n(n-1)!$.

10. (a) Stanley, Exercise 5.5. A proper n-coloring of a graph $G = (V, E)$ is a function $c : V \to [n]$ such that $c(v) \neq c(w)$ whenever v, w are the endpoints of an edge $e \in E$. A graph G is *bipartite* if there exists a proper 2-coloring of G.

(b) Let B be the species $B(S) = \{$bipartite graphs with vertex set $S\}$ and let G_2 be the species $G_2(S) = \{$graphs G with vertex set S, together with a proper 2-coloring of $G\}$. Part (a) implies that G_2 and B^2 have the same exponential generating function, and indeed the same mixed generating function weighted by number of edges. Are the species B^2 and G_2 equivalent?

(c) Derive the mixed ordinary/exponential generating function for the species of *connected* bipartite graphs, weighted by $q^{\text{number of edges}}$. Use it to calculate the number of connected labelled bipartite graphs with k edges on n vertices for $n \leq 5$ and all k.

11. (a) The *diameter* d of a tree T is the maximum length of a path in T (a path with n vertices has length $n-1$). Prove that if d is even then all paths of length d have the same middle vertex, called the *center* of T, and if d is odd, then all paths of length d have the same middle edge; the path of length 1 consisting of this edge and its two endpoints is called the *bicenter* of T.

(b) Show that if d is even, the species of labelled unrooted trees of diameter d is equivalent to the species of labelled rooted trees of height $d/2$ with the property that at least two children of the root are roots of subtrees of height $d/2 - 1$.

(c) Show that if d is odd, the species of labelled unrooted trees of diameter d is equivalent to the species of unordered pairs of disjoint rooted trees of height $(d-1)/2$.

(d) Let T_h be the species of labelled rooted trees of height h, and let $T_{\leq h} = T_0 + \cdots + T_h$. Show that these are given by the recurrence

$$\begin{aligned} &T_0 = X \\ &T_h = X((E-1)\circ T_{h-1})(E \circ T_{\leq h-2}) \quad \text{for } h > 0. \end{aligned}$$

(e) Using products and composition of species, express the species of labelled unrooted trees of diameter d in terms the species T_h for various h.

(f) Use the above to calculate the number of labelled unrooted trees of diameter d on n vertices, for $n \leq 5$ and all d. Check your answer by summing over d.

Lecture 9

1. (a) Show that the exponential generating function for the species of labelled unrooted forests is given by

$$F(x) = \exp \sum_{n=1}^{\infty} n^{n-2} \frac{x^n}{n!}.$$

(b) Use (a) to calculate the number of labelled unrooted forests on $[n]$ for $n \leq 6$.

(c) Modify (a) to get a mixed ordinary/exponential generating function $F(t,x)$ for the species of labelled unrooted forests, in which the coefficient of t^k forest with k components. Use this to refine your answer to (b) to count forests by number of components.

2. A *leaf-labelled* tree T is a rooted tree whose leaves are the elements of a given set S, and whose other nodes are unlabelled. Let us require each non-leaf node of T to have at least two children; then there are finitely many leaf-labelled trees on a given finite leaf set S.

Introduce indeterminates $a_2, a_3, \ldots$ and assign each leaf-labelled tree the weight $w(T) = \prod_i a_i^{r_i}$, where r_i is the number of non-leaf nodes in T with i children. Then show that the mixed ordinary/exponential generating function enumerating leaf-labelled trees with these weights is the functional composition inverse of $x - A(x)$, where $A(x) = \sum_{n \geq 2} a_n x^n / n!$.

3. Use Cayley's formula to calculate that the number of strictly binary (*i.e.*, rooted with every non-leaf node having exactly two children) unlabelled ordered trees on $2n+1$ vertices (such a tree always has an odd number of vertices) is the Catalan number C_n.

4. Prove that the number of ordered rooted trees with $n+1$ vertices and j leaves is equal to

$$\frac{1}{n+1}\binom{n+1}{j}\binom{n-1}{n-j}.$$

5. Prove that the number of ways to subdivide an n-gon into k polygons by introducing $k-1$ diagonals that do not intersect except at their endpoints is equal to the number of ordered rooted trees with $n+k-1$ vertices, $n-1$ leaves, and no vertices with exactly one child. Derive the formula

$$\frac{1}{n+k-1}\binom{n+k-1}{n-1}\binom{n-3}{n-k-2}$$

for this number. In particular, taking $k = n-2$, deduce that the number of triangulations of an n-gon is the Catalan number C_{n-2}. In this problem the n-gon is regarded as fixed in place, so for example the two triangulations of a square count as different even though they are the same up to symmetry.

Lectures 10-11

1. Given a formal series $E(x) = e_0 + e_1 x + e_2 x^2 + \cdots$, where e_0 is invertible, let $F(x)$ be the functional composition inverse of $G(x) = x/E(x)$, *i.e.*, $F(x)$ satisfies the functional equation

$$F(x) = xE(F(x)).$$

We saw in the lecture that the coefficient $\langle x^n \rangle F(x)$ is the ordinary generating function enumerating (unlabelled) ordered rooted trees (Stanley calls them *plane trees*) on n vertices, with weight

$$\prod_{v \in T} e_{d(v)},$$

where $d(v)$ is the number of children of vertex v in T. We also gave a combinatorial proof of the case $k = 1$ of the following more general form of the Lagrange inversion formula:

$$\langle x^{n+k} \rangle F(x)^k = \frac{k}{n+k} \langle x^n \rangle E(x)^{n+k}. \tag{1}$$

(a) Show that for any sequence $(r_1, \ldots, r_{n+k}) \in \mathbb{N}^{n+k}$ such that $r_1 + \cdots + r_{n+k} = n$, exactly k of its rotations $(s_1, \ldots, s_{n+k}) = (r_{l+1}, \ldots, r_{n+k}, r_1, \ldots, r_l)$ satisfy

$$s_1 + \cdots + s_i + k > i \quad \text{for all } 0 \le i < n+k.$$

(b) Show that

$$\frac{k}{n+k} \sum_{r_1 + \cdots + r_{n+k} = n} e_{r_1} \cdots e_{r_{n+k}} = \sum_{\lambda \subseteq (n+k-2, \ldots, k-1)} \prod_{i=0}^{n+k-1} e_{\alpha_i(\lambda)},$$

where λ ranges over partitions whose Young diagram fits inside that of $(n+k-2, \ldots, k-1)$, and $\alpha_i(\lambda)$ is the number of parts equal to i in λ, with $\alpha_0(\lambda)$ defined so that $\sum_i \alpha_i(\lambda) = n$.

(c) Use (b) to give a combinatorial proof of the generalized Lagrange inversion formula above.

2. In the lecture we discussed a q-analog of functional composition defined by

$$F \circ_q G(x) = \sum_k f_k \, G(x) G(qx) \cdots G(q^{k-1} x),$$

where $G(x)$ is a formal series without constant term and q is an element of the ground ring.

(a) Show that the operator Ψ on formal power series defined by $\Psi(F) = F \circ_q G$ is continuous, linear over the ground ring and satisfies $\Psi(1) = 1$ and $\Psi(x\, F) = G(x)\, (\Psi(F)(qx))$.

(b) Prove that the operator Ψ is determined by the properties in (a).

(c) Assume now that both q and the coefficient of the linear term of $G(x)$ are invertible in the ground ring. Prove that the operator Ψ has an inverse.

(d) Let $H = \Psi^{-1}(x)$, that is, $H \circ_q G = x$. Prove the identity $\Psi(HF) = x\Psi(F(qx))$, for all F.

(e) Prove Garsia's Theorem, which states that Ψ^{-1} is given by $\Psi^{-1}(F) = F \circ_{1/q} H$. In particular, deduce that $G \circ_{1/q} H = x$.

Lecture 12

1. A rooted forest is a disjoint union of rooted trees. Prove the identity

$$\sum_F \prod_j x_j^{c_F(j)} = \binom{n-1}{k-1}(x_1 + \cdots + x_n)^{n-k},$$

where F ranges over rooted forests with k components on vertex set set $[n]$. In particular, the number of such forests is $\binom{n-1}{k-1}n^{n-k}$.

2. Let $f_m(r)$ be the number of rooted spanning forests with r roots in the graph C_m, a cycle on m vertices ($m > 1$). Prove that

$$F_m(z) \underset{\text{def}}{=} \sum_r f_m(r)z^r = \prod_{j=0}^{m-1}(z + 2 - 2\cos(2\pi j/m)) = \sum_{r=1}^{m} \frac{m}{r}\binom{m+r-1}{2r-1} z^r.$$

3. The product $G \times H$ of two simple graphs (graphs without loops or multiple edges) is the graph on vertex set $V(G) \times V(H)$ with edges $\{(v,w),(v',w')\}$ for $v = v'$ and $\{w,w'\} \in E(H)$ or $w = w'$ and $\{v,v'\} \in E(G)$. The adjacency matrix A_G of a graph G on n vertices is the $n \times n$ matrix with rows and columns labelled by the vertices, and entries $(A_G)_{v,w} = 1$ if $\{v,w\} \in E(G)$, zero otherwise. Let D_G be the diagonal matrix whose (v,v) entry is the degree of v.

(a) Let $f_G(r)$ be the number of rooted spanning forests of G with r roots, and let $F_G(z) = \sum_r f_G(r)z^r$ be the corresponding generating function. Show that $F_G(z) = \prod_i (z+\alpha_i)$, where the α_i's are the eigenvalues of $D_G - A_G$.

(b) Show that $F_{G\times H}(z) = \prod_{i,j}(z + \alpha_i + \beta_j)$, where $F_G(z) = \prod_i (z + \alpha_i)$ and $F_H(z) = \prod_j (z+\beta_j)$. In particular, the numbers $f_G(r)$ and $f_H(r)$ for all r determine the corresponding numbers $f_{G\times H}(r)$.

(c) Show that if Q_n is the graph formed by the vertices and edges of the n-cube, that is, the product of n copies of the complete graph on 2 vertices, then

$$F_{Q_n}(z) = \prod_{k=0}^{n} (z + 2k)^{\binom{n}{k}}.$$

This generalizes Stanley, Exercise 5.6.10, which follows by taking the coefficient of z.

Lectures 13-14

1. Two parts:

(a) Verify by direct calculation that the cycle index Z_C for the species of cyclic orderings (*i.e.*, permutations with one cycle) is given by

$$Z_C = \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \frac{1}{1-p_n},$$

where ϕ is Euler's totient function: $\phi(n)$ is the number of integers $r \in [n]$ relatively prime to n.

(b) Check that $Z_C(x, 0, \ldots)$ and $Z_C(x, x^2, \ldots)$ agree, respectively, with the exponential generating function for the species of cyclic orderings, and the ordinary generating function for cyclic orderings up to isomorphism.

2. Recall that the cycle index of the trivial species is given by

$$Z_E = \exp \sum_{n=1}^{\infty} p_n/n.$$

Verify that the plethysm $Z_E * Z_C$ agrees with the formula we obtained by direct calculation for the cycle index of the species of permutations,

$$Z_P = \prod_{n=1}^{\infty} \frac{1}{1-p_n}.$$

Lecture 15

1. For $1 \leq i < j \leq n$, define the raising operator R_{ij} on $\mathbb{Z}^n$ by

$$R_{ij}(\nu_1, \ldots, \nu_n) = (\nu_1, \ldots, \nu_i + 1, \ldots, \nu_j - 1, \ldots, \nu_n).$$

(a) Show that the dominance order $\leq$ is the transitive closure of the relation on partitions $\lambda \to \mu$ if $\mu = R_{ij}\lambda$ for some $i < j$.

(b) We say that μ *covers* λ if $\lambda < \mu$ and there is no ν such that $\lambda < \nu < \mu$. Show that μ covers λ if and only if $\mu = R_{ij}\lambda$, where i, j satisfy the following condition: either $j = i+1$, or $\lambda_i = \lambda_j$ (or both).

(c) Find the smallest n such that the dominance order on partitions of n is not a total ordering, and draw its Hasse diagram (i.e., the graph of the covering relation).

2. Express $m_\lambda(1, 1, \ldots, 1)$, with n ones, as a more familiar combinatorial quantity.

3. (a) Use the fundamental theorem of symmetric functions to show that if $f(t) = t^k + a_1 t^{k-1} + \cdots + a_k$ and $g(t) = t^l + b_1 t^{l-1} + \cdots + b_l$, there is a polynomial $R_{f,g}(a_1, \ldots, a_k, b_1, \ldots, b_l)$ such that $R_{f,g} = 0$ if and only if f and g have a common root. The minimal such polynomial (which is unique up to a constant factor) is called the *resultant* of f and g. Calculate $R_{f,g}$ for $k = 2$ and $l = 3$.

(b) Show that $R_{f,g}$ is the determinant of the $(k+l) \times (k+l)$ matrix

$$\begin{bmatrix}
1 & a_1 & \ldots & a_l & 0 & \ldots & 0 & 0 \\
0 & 1 & a_1 & \ldots & a_l & 0 & \ldots & 0 \\
 & & & & \vdots & & & \\
0 & 0 & \ldots & 0 & 1 & a_1 & \ldots & a_l \\
1 & b_1 & \ldots & b_k & 0 & \ldots & 0 & 0 \\
0 & 1 & b_1 & \ldots & b_k & 0 & \ldots & 0 \\
 & & & & \vdots & & & \\
0 & 0 & \ldots & 0 & 1 & b_1 & \ldots & b_k
\end{bmatrix}$$

Lecture 16

1. Prove the formula for complete homogeneous symmetric functions in terms of elementary symmetric functions

$$h_n = \sum_{|\lambda|=n} (-1)^{n-l(\lambda)} \binom{l(\lambda)}{r_1, r_2, \ldots, r_k} e_\lambda,$$

where $\lambda = (1^{r_1}, 2^{r_2}, \ldots, k^{r_k})$.

2. The symmetric functions $f_\lambda = \omega m_\lambda$ are sometimes called the "forgotten" symmetric functions. Show that the matrix of coefficients of the forgotten symmetric functions f_λ expressed in terms of monomial symmetric functions m_λ is the transpose of the matrix of the elementary symmetric functions e_λ expressed in terms of the complete homogeneous symmetric functions h_λ.

3. From Macdonald's book:

(a) Recall from class that $h_n = \sum_{|\lambda|=n} p_\lambda / z_\lambda$, where $z_\lambda = \prod_i i^{r_i} r_i!$ for $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$. Show that this is equivalent to Newton's determinant formula

$$h_n = \frac{1}{n!} \det \begin{bmatrix} p_1 & -1 & 0 & \ldots & 0 \\ p_2 & p_1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n-1} & p_{n-2} & . & \ldots & -(n-1) \\ p_n & p_{n-1} & . & \ldots & p_1 \end{bmatrix}$$

(b) Show that e_n is given by the same determinant without the minus signs.

Lectures 17-18

1. For any symmetric polynomial f, let $f^\perp$ be the operator adjoint to multiplication by f with respect to the Hall inner product, that is, $\langle f^\perp g, h\rangle = \langle g, fh\rangle$ for all g, h.

(a) Find a formula for $h_k^\perp m_\lambda$, expressed again in terms of monomial symmetric functions m_μ.

(b) Show that the basis of monomial symmetric functions is uniquely characterized by the formula from part (a).

2. Let ∂p_k be the operator on symmetric functions given by partial differentiation with respect to p_k, under the identification of the algebra of symmetric functions with the polynomial ring $\mathbb{Q}[p_1, p_2, \ldots]$. Show that ∂p_k is adjoint with respect to the Hall inner product to the operator of multiplication by p_k/k.

Lecture 19

1. If A is an algebra over a field k, then $A \otimes_k A$ is an algebra with multiplication characterized uniquely by $(a \otimes b)(c \otimes d) = ac \otimes bd$. A *coproduct* is an algebra homomorphism $\Delta\colon A \to A \otimes A$ which is *coassociative* in the sense that the two maps $(1 \otimes \Delta) \circ \Delta$ and

$(\Delta \otimes 1) \circ \Delta$ from A to $A \otimes A \otimes A$ are equal. This axiom is dual to the associative law for multiplication $\mu\colon A \otimes A \to A$, which can be formulated as $\mu \circ (1 \otimes \mu) = \mu \circ (\mu \otimes 1)$. Here $1\colon A \to A$ denotes the identity map.

Taking Λ to be the algebra of symmetric functions with coefficients in $k = \mathbb{Q}$, and identifying $\Lambda \otimes \Lambda$ with $\Lambda(X)\Lambda(Y)$, show that $\Delta(f) = f[X+Y]$ defines a coproduct on Λ.

Remark: An algebra equipped with a coproduct is called a *bialgebra*. If we also define the *counit* $\epsilon\colon \Lambda \to \mathbb{Q}$ by $\epsilon(f) = \langle f, 1\rangle = f[0]$ and the *antipode* $S\colon \Lambda \to \Lambda$ by $Sf = f[-X]$, these together with Δ can be shown to satisfy the axioms of a *Hopf algebra*.

2. Consider an alphabet $\mathcal{A} = \{x_1, y_1, x_2, y_2, \ldots\}$ of two kinds of variables. Fix any ordering of $\mathcal{A}$ and define the "super" Schur function $s_\lambda(x; y)$ to be the generating function for "super" semistandard Young tableau of shape λ. Such a tableau is a filling of the diagram of λ by variables from $\mathcal{A}$, weakly increasing along rows and columns as usual, with the requirement that no x_i is repeated in any column and no y_i is repeated in any row. In particular, $s_\lambda(x; 0) = s_\lambda(x)$ and $s_\lambda(0; y) = s_{\lambda'}(y)$.

(a) Prove that $s_\lambda(x; y)$ is symmetric in the x_i's and the y_i's separately, and does not depend on the ordering chosen for $\mathcal{A}$.

(b) Prove that if $y_1 = x_1$, then $s_\lambda(x_1, x_2, \ldots; -y_1, -y_2, \ldots) = s_\lambda(x_2, \ldots; -y_2, \ldots)$.

(c) Prove that, in plethystic notation,

$$s_\lambda(x; -y) = s_\lambda[X - Y].$$

(d) Prove that

$$s_{(n^m)}(x_1, \ldots, x_n; -y_1, \ldots, -y_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x_i - y_j).$$

(e) Prove that the resultant $R_{f,g}$ from Lecture 15, Problem 3 is given by the expansion of $s_{(k^l)}[X - Y]$ in terms of elementary symmetric functions $e_j[X]$ and $e_j[Y]$, where these are equated with the coefficients of f and g by the rule $a_j = (-1)^j e_j[X]$, $b_j = (-1)^j e_j[Y]$.

3. Let ϵ be a fictitious alphabet such that $p_k[\epsilon] = \delta_{1,k}$. Stated more correctly, this means we are to interpret $f[\epsilon]$ as the image of f under the homomorphism $\Lambda \to \mathbb{Q}$ mapping p_k to $\delta_{1,k}$.

(a) Prove the identity $f[\epsilon] = \langle f, \exp(p_1)\rangle$.

(b) Prove the identity $f[\epsilon] = \lim_{n\to\infty} (f[nx])_{x \mapsto 1/n}$.

(c) Show that $e_k[\epsilon] = h_k[\epsilon] = 1/n!$.

(d) More generally, show that $s_\lambda[\epsilon] = f_\lambda/n!$, where $|\lambda| = n$ and f_λ is the number of standard Young tableaux of shape λ.

Lecture 20

1. From Macdonald's book: prove the identity $s_{(n-1,n-2,\ldots,1)}(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} (x_i + x_j)$.

2. Let λ be a partition whose diagram has m cells on the main diagonal. Let α_i be the number of cells to the right of the i-th diagonal cell, and β_i be the number above it. The Frobenius notation for λ is the sequence $(\alpha_1, \ldots, a_m|\beta_1, \ldots, \beta_m)$. For example, the partition $(6, 4, 2, 2)$ is denoted $(5, 2|3, 2)$ in Frobenius notation. Let $s_{(a|b)}(x) = s_{(a+1,1^b)}(x)$ be the Schur function corresponding to a partition of "hook" shape.

Prove the *Giambelli formula*

$$s_{(\alpha|\beta)} = \det\left[s_{(\alpha_i|\beta_j)}\right]_{1\leq i,j\leq m}.$$

Hints:

(a) Show that $s_{(a|b)} = h_{a+1}e_b - h_{a+2}e_{b-1} + \cdots$, and extend its definition to all integers a, b by this formula, with the usual convention that $h_k = e_k = 0$ for $k < 0$. Next show that $s_{(a|b)} = (-1)^b$ for $b \geq 0$ and $a = -b-1$, and that $s_{(a|b)} = 0$ in all other cases when a or b is negative.

(b) Choose $M \geq \max(\lambda_1, l(\lambda))$ and let H be the matrix

$$H = \left[h_{\lambda_i+j-i}\right]_{1\leq i,j\leq M},$$

so $\det(H) = s_\lambda$. Let E be the matrix

$$E = \left[(-1)^{i-1}e_{j-i}\right]_{1\leq i,j\leq M}.$$

Let S be the matrix in the Giambelli formula. Show that there is a permutation matrix P such that $HE = XP$, where X is the $M \times M$ block diagonal matrix

$$X = \begin{bmatrix} S & 0 \\ 0 & I_{M-m} \end{bmatrix}.$$

Lecture 21

1. Show that G is abelian if and only if all the irreducible representations of G are one-dimensional.

2. Use Maschke's theorem to prove that if a square matrix A over $\mathbb{C}$ satisfies $A^m = I$ for some m, then A is diagonalizable. More generally, given several matrices A_i such that $A_i^{m_i} = I$ and $A_iA_j = A_jA_i$ for all i, j, prove that they are simultaneously diagonalizable.

3. Find the character tables of

(a) the dihedral group D_8 of order 8;

(b) the quaternion group $Q_8 = \{\pm1, \pm i, \pm j, \pm k\}$, where $i^2 = j^2 = k^2 = ijk = -1$, and signs multiply according to the usual rules.

Show that Q_8 and D_8 are not isomorphic, and conclude that the character table of a finite group need not determine the group.

4. Show that all the characters of G are real if and only if g is conjugate to g^{-1} for all $g \in G$.

5. Let k be the finite field with p elements. The upper unit-triangular 2×2 matrices over k form a matrix representation of $G = \mathbb{Z}/p\mathbb{Z}$. Show that the corresponding G-module V is indecomposable but not irreducible, providing a counterexample to Maschke's theorem over a field of prime characteristic.

Lecture 22

1. From Macdonald's book: let $|\lambda| = |\mu| = n$. Show that $\langle h_\lambda, h_\mu \rangle$ is equal to the number of double cosets $S_\lambda w S_\mu$ in the symmetric group S_n, where S_λ and S_μ are Young subgroups of S_n.

2. Compute the character table of S_5.

3. Let $K_{\lambda\mu} = \langle s_\lambda, h_\mu \rangle$ be the number of semistandard Young tableaux of shape λ and content μ. Show that $K_{\lambda\mu}$ is equal to the dimension of the space of invariants $V_\lambda^{S_\mu}$, where V_λ is the irreducible representation of S_n indexed by the partition λ, and S_μ is the Young subgroup $S_{\mu_1} \times \cdots \times S_{\mu_l} \subseteq S_n$.

4. Prove that the Frobenius characteristic map is given by

$$F\chi_V = \sum_\mu \dim(V^{S_\mu}) m_\mu,$$

where V is an S_n module and S_μ is as in the preceding problem.

5. Two parts:

(a) Prove that $V_{(n-1,1)}$ is the irreducible submodule of dimension of dimension $n-1$ in the defining representation of S_n on $\mathbb{C}^n$.

(b) Prove that $V_{(n-k,1^k)}$ is isomorphic to the k-th exterior power of $V_{(n-1,1)}$.

6. Describe explicitly a 5-dimensional irreducible representation $\rho\colon S_5 \to GL_5$, by giving matrices $\rho(\sigma)$ for some elements σ that generate S_5. [Labor-saving hint: S_n can be generated by two elements.]

Lectures 23-25

1. If V is a G-module and H is a subgroup of G, then H acts on V, so we can consider V as an H-module, called the *restriction* of V to H, and denoted $V|_H$. For this problem we will take $G = S_n$ and $H = A_n$, the alternating group. Denote by V_λ the irreducible representation of S_n whose character χ_λ corresponds to s_λ via the Frobenius characteristic map.

(a) Show that $V_\lambda|_{A_n} \cong V_{\lambda'}|_{A_n}$.

(b) Show that $V_\lambda|_{A_n}$ is irreducible if $\lambda \neq \lambda'$, and that it is the direct sum of two inequivalent irreducible representations if $\lambda = \lambda'$. Also show that the irreducible constituents of $V_\lambda|_{A_n}$ are not isomorphic to those of $V_\mu|_{A_n}$ if $\{\lambda, \lambda'\} \neq \{\mu, \mu'\}$. [Hint: relate the character χ of $V_\lambda|_{A_n}$ to the S_n character $\chi_\lambda + \chi_{\lambda'}$.]

(c) Describe the restriction of the regular representation of S_n to A_n. Deduce that every irreducible representation of A_n occurs in the restriction of some irreducible representation of S_n.

(d) Show that each conjugacy class of even permutations in S_n is a conjugacy class in A_n, with the following exceptions: a class consisting of permutations whose cycles have distinct, odd lengths splits into two conjugacy classes in A_n.

(e) Let $p(n)$ be the number of partitions of n, $p_{\mathrm{ee}}(n)$ the number with an even number of even parts, and $k(n)$ the number of self-conjugate partitions, *i.e.*, such that $\lambda = \lambda'$. Recall that $k(n)$ is also equal to the number of partitions of n with distinct, odd parts. Show that the result $p_{\mathrm{ee}}(n) = (p(n)+k(n))/2$, previously obtained from partition generating functions, is equivalent to the special case for the group A_n of the equality between the number of irreducible characters and the number of conjugacy classes.

2. Compute the character table of A_5.

3. Three parts:

(a) Let V be a representation of G with character χ. Prove that $g \in G$ acts trivially on V if and only if $\chi(g^k) = \chi(1)$ for all k.

(b) Show that every proper normal subgroup $H \subseteq G$ acts trivially on some non-trivial irreducible representation of G.

(c) Read off a proof that A_5 is simple from its character table.

Lecture 26

1. Prove that standard tableaux S and T are dual equivalent if and only if there exist standard tableaux S' and T' of some straight shape λ and a tableau X (of a shape ν for which $\lambda \sqcup \nu$ makes sense) such that $J^X(S') = S$ and $J^X(T') = T$.

2. Two parts:

(a) Let S and T be standard tableau of the same (skew) shape ν, where $|\nu| = 3$. Verify directly that if there exists a slide into the same cell that gives tableaux of different shapes when applied to S and T, then the reading words of S and T must differ by a switch of entries in adjacent positions in the word. In particular, they cannot be $\{213,\ 312\}$ or $\{132,\ 231\}$.

(b) Use part (a) and the fact that jeu-de-taquin preserves descent sets to show that if if S and T have reading words $\{213,\ 312\}$ or $\{132,\ 231\}$, then any slide into the same cell applied to S and T yields another pair of tableaux S', T' with the same shape and reading words $\{213,\ 312\}$ or $\{132,\ 231\}$. Deduce that any such pair is dual equivalent. [This avoids the case checking needed for the proof indicated in the lecture.]

3. Let P be a partially ordered set with a least element $\mathbf{0}$. Assume P *locally finite*, which means that every interval $[\mathbf{0}, x]$ is finite. Define a *shape* to be a finite subset $\nu \subseteq P$ such that $x \le y \le z$ and $x, z \in \nu$ imply $y \in \nu$; define a *standard tableau* of shape ν to be an order-preserving bijection $\nu \to [n]$, where $n = |\nu|$. If P is $\mathbb{N} \times \mathbb{N}$, these reduce to the usual notions of (skew) shapes and tableaux. Define foward jeu-de-taquin slides for tableaux on P analogously to the definition for $P = \mathbb{N} \times \mathbb{N}$.

One says that P has the *jeu-de-taquin property* if for every tableau T and every sequence of forward slides that carries T into a shape containing $\mathbf{0}$, the resulting tableau S depends only on T and not on the sequence of slides chosen. The fundamental theorem of jeu-de-taquin states that $P = \mathbb{N} \times \mathbb{N}$ has the jeu-de-taquin property. Show that the following posets have the jeu-de-taquin property.

(a) Trees (this is easy).

(b) The subset $R_k \subseteq \mathbb{Z} \times \mathbb{Z}$ which is the union of $\mathbb{N} \times \mathbb{N}$ and the set $\{(1, -1), (0, -1), (0, -2), \ldots, (0, -k)\}$.

(c) The subset $Q = \{(i, j) : i \leq j\} \subseteq \mathbb{N} \times \mathbb{N}$. This poset is called the *shifted plane*. Hint: set up a theory of dual equivalence for tableaux on Q in which the elementary dual equivalences involve tableaux of size 4. You can find them all by starting with the unique shape of size 4 that contains $\mathbf{0}$ and posseses two distinct tableaux.

[All of the above are special cases of Proctor's notion of *d-complete* posets (see J. Algebra 213 (1999), 272–303; J. Alg. Combinatorics 9 (1999), 61–94.]

4. Let $a < b < c$ be consecutive entries in a standard tableau T. Suppose that in the word of T (reading row by row from top to bottom in French style notation) they occur in one of the orders bac, cab, acb, or bca. Then we can always switch the outer two of these three entries in T to get another standard tableau S (verify). By a lemma in the lecture, S and T are dual equivalent. We call this situation an *elementary* dual equivalence.

(a) Prove that dual equivalence is the transitive closure of elementary dual equivalence.

(b) Deduce that tableaux S and T of the same shape are dual equivalent if and only if their reading words are connected by a sequence of "dual Knuth relations" of the form $bac \leftrightarrow cab$ or $acb \leftrightarrow bca$, where $a < b < c$ are consecutive and appear in the specified order in the reading word, but not necessarily in adjacent positions.

5. Call a skew shape *anti-straight* if it is a 180° rotation of a straight (non-skew) shape. Equivalently, a skew shape is anti-straight if it can be written as λ/μ, where λ is a rectangle.

(a) Show that if X has anti-straight shape and $T \sqcup X$ makes sense, then $J^X(T)$ has anti-straight shape and depends only on T, not on X. Denote this tableau by $J^{\square}(T)$.

(b) Show that if T has straight shape λ, then the shape of $J^{\square}(T)$ is the 180° rotation of λ. [Hint: using dual equivalence, it suffices to do this for one specially chosen T of the given shape.]

(c) Schützenzerger's *evacuation* operator $T \to \mathrm{ev}(T)$ is defined as follows. Given T of straight shape, $\mathrm{ev}(T)$ is the tableau obtained from $J^{\square}(T)$ by rotating it 180° and renumbering the entries in reverse order, $1, 2, \ldots, n \mapsto n, \ldots, 2, 1$. Show that evacuation is an involution.

6. Consider the following sequence of operations on a tableau T of straight shape λ:

(1) delete the smallest entry;
(2) perform a jeu-de-taquin slide into the now empty cell $(0, 0)$;
(3) repeat steps (1) and (2) until all entries have been removed;
(4) form the tableau S whose entries $n, \ldots, 2, 1$ occupy the cells of λ in the order they are vacated by step (2).

Show that the resulting tableau S is equal to the evacuation $\mathrm{ev}(T)$.

Lecture 27

1. Use the adjointness between multiplication and substitution $f \mapsto f[X+Y]$ to prove that the Schur function attached to a skew diagram λ/μ satisfies (and is determined by) the identity

$$\langle s_{\lambda/\mu}, g\rangle = \langle s_\lambda, s_\mu g\rangle.$$

2. Consider the partition (a^b) whose diagram is an $a \times b$ rectangle. Prove that

$$s_{(a^b)}^2 = \sum_\mu s_\mu,$$

where all coefficients are equal to 1 and the sum ranges over partitions μ containing (a^b) and such that $\mu/(a^b)$ is the disjoint union of (translates of) partition diagrams $\rho, \nu \subseteq (a^b)$, satisfying $\nu = ((a^b)/\rho)^\perp$. Here $(-)^\perp$ denotes rotation of a diagram through $180°$.