Lecture 1

1. Let $\langle x^n \rangle F(x)$ denote the coefficient of $x^n$ in a formal power series $F(x)$. A sequence $(F_k(x))$ in $R[[x]]$ is said to converge to $G(x)$ if, for each $n$, the sequence $(\langle x^n \rangle F_k(x))$ converges to $\langle x^n \rangle G(x)$ in the discrete topology on $R$; that is, if $\langle x^n \rangle F_k(x) = \langle x^n \rangle G(x)$ for all sufficiently large $k$. The definition adapts in an obvious way to formal power series in several variables.

(a) Show that the partial sums $f_0 + f_1 x + \cdots + f_k x^k$ of $F(x)$ converge to $F(x)$.

(b) Show that a sum $\sum_{k=1}^{\infty} F_k(x)$ converges if and only if $(F_k(x))$ converges to 0. Show that rearranging the terms does not change the property of convergence, nor the value of the limit.

(c) Show that a product $\prod_{k=1}^{\infty} F_k(x)$ converges if $(F_k(x))$ converges to 1; if $R$ is an integral domain, show that this is if and only if. Again show that the limit and the property of convergence are independent of rearranging the terms.

(d) Show that a sum or product of limits of convergent sequences is the limit of the sums or products term by term. Show that this also holds for convergent infinite sums and products.

2. If $F(x)$ and $G(x)$ are formal power series, and $F(0) = 0$ (where $F(0)$ is defined to be the constant term of $F(x)$), their formal composition is defined by

$$(G \circ F)(x) = \sum_{k=0}^{\infty} g_k F(x)^k,$$

where $G(x) = \sum_{k=0}^{\infty} g_k x^k$. Note that the sum converges by Problem 1(b). We also write $G(F(x))$ for $(G \circ F)(x)$.

(a) Show that composition is associative, i.e., if $F(0) = 0$ and $G(0) = 0$, then $(H \circ G) \circ F = H \circ (G \circ F)$.

(b) Show that composition with a fixed $F$, considered as a function of $G$, is a ring homomorphism.

(c) Show that $F(x) \in R[[x]]$ such that $F(0) = 0$ has a formal compositional inverse if and only if the coefficient $\langle x \rangle F(x)$ has a multiplicative inverse in $R$.

3. Show that $F(x) \in R[[x]]$ has a multiplicative inverse if and only if its constant term $F(0)$ has a multiplicative inverse in $R$.

4. Assume we are working in a formal power series ring $R[[x]]$ over a coefficient ring $R$ containing $\mathbb{Q}$. Define $\exp(x)$ and $\log(1 + x)$ by their usual Taylor series.

(a) Given $F(x), G(x) \in R[[x]]$ such that $F(0) = 1$, show that there is a well-defined formal power series

$$F(x)^{G(x)} \overset{\text{def}}{=} \exp(G(x) \log F(x)).$$

(b) Show that the above definition satisfies the usual laws of exponents, namely, $F(x)^{G(x)+H(x)} = F(x)^{G(x)} F(x)^{H(x)}$, $F(x)^{G(x)H(x)} = (F(x)^{G(x)})^{H(x)}$, $F(x)^0 = 1$, $F(x)^1 = F(x)$. 

Homework problems
5. Defining the derivative \( \frac{d}{dx} F(x) \) of a formal power series in the obvious formal way, show that the usual sum and product rules and the chain rule hold. If the coefficient ring contains \( \mathbb{Q} \), show that Taylor’s formula holds.

6. (a) Give two proofs of the binomial coefficient identity, called the convolution formula,
\[
\sum_j \binom{m}{j} \binom{n}{k-j} = \binom{m+n}{k}.
\]
One proof should use generating functions, the other should be a direct combinatorial proof.

(b) Discover and prove in the same two ways an analogous identity for multiset coefficients \( \langle \binom{n}{k} \rangle \).

7. Find a simple expression for the ordinary generating function in two variables
\[
\sum_{n,k \geq 0} \binom{n}{k} x^n y^k,
\]
and use it to deduce the identity
\[
\sum_r \binom{r}{k} x^r = \frac{x^k}{(1-x)^{k+1}}.
\]

8. Fix \( k \) and let \( a(n, k) \) be the number of words \( W \) of length \( n \) with entries in \([k]\), such that \( W \) contains all \( k \) symbols at least once, and for each \( j \), all the symbols \( 1, \ldots, j - 1 \) appear at least once to the left of the first \( j \). Find the ordinary generating function
\[
A_k(x) = \sum_n a(n, k)x^n.
\]

9. Fix \( n \) and let \( c(n, k) \) be the number of permutations of \([n]\) with \( k \) cycles. Prove the ordinary generating function identity
\[
\sum_k c(n, k)y^k = y(y + 1) \cdots (y + n - 1).
\]

Lecture 2

Homework for Lectures 1 and 2 due Jan. 31.
1. Stanley, Ch. 1, Ex. 8(a).
2. Stanley, Ch. 1, Ex. 31.
3. [corrected 1/31] Prove that the \( q \)-multinomial coefficients satisfy the recurrence
\[
\binom{n}{k_1, k_2, \ldots, k_r}_q = \binom{n-1}{k_1-1, k_2, \ldots, k_r}_q + q^{k_1} \binom{n-1}{k_1, k_2-1, \ldots, k_r}_q + \cdots + q^{k_1+\cdots+k_{r-1}} \binom{n-1}{k_1, k_2, \ldots, k_{r-1}-1}_q.
\]
4. Prove the following $q$-analog of the convolution formula for binomial coefficients.

\[
\binom{m+n}{k}_q = \sum_{i+j=k} q^{(m-i)j} \binom{m}{i}_q \binom{n}{j}_q
\]

5. Let $l$ divide $n$. Show that the primitive $l$-th roots of unity are roots of the polynomial $\binom{n}{k}_q$ in $q$ if and only if $l$ does not divide $k$.

6. Regarding \( \binom{x}{k} \) as a polynomial of degree $k$ in the variable $x$, prove that a polynomial $f \in \mathbb{Q}[x]$ has the property that $f(q^n)$ is an integer for all integers $n$ if and only if the coefficients of $f$ with respect to the basis \( \{ \binom{x}{k} : k \in \mathbb{N} \} \) are integers.

   Hint: one direction is easy. For the other, use the fact that $0, 1, \ldots, k - 1$ are roots of $(x)_k$, and $(\binom{k}{k}) = 1$.

7. (a) Show that for each $k$ there is a unique polynomial $Q_k(x)$ of degree $k$, with coefficients in the field of rational functions $\mathbb{Q}(q)$, such that $Q_k(q^n) = \binom{n}{k}_q$ for all $n$.

   (b) Prove that a polynomial $f \in \mathbb{Q}(q)[x]$ has the property that $f(q^n) \in \mathbb{Z}[q, 1/q]$ for all $n$ if and only if the coefficients of $f$ with respect to the basis \( \{ Q_k : k \in \mathbb{N} \} \) belong to $\mathbb{Z}[q, 1/q]$.

Lecture 3

1. Prove Cauchy’s identity:

\[
\prod_{i \geq 0} \frac{1 - axq^i}{1 - xq^i} = \sum_{n \geq 0} \frac{(1 - a)(1 - aq) \cdots (1 - aq^{n-1})}{(1 - q)(1 - q^2) \cdots (1 - q^n)} x^n,
\]

by showing that it reduces to the $q$-binomial theorem upon setting $a = q^m$ for an integer $m$ (note that you get either form of the $q$-binomial theorem by taking $m$ positive or negative).

2. (a) Give a direct combinatorial proof of the partition identities

\[
\prod_{i \geq 1} \frac{1}{1 - xq^i} = \sum_{n \geq 0} \frac{x^n q^n}{(1 - q)(1 - q^2) \cdots (1 - q^n)}
\]

and

\[
\prod_{i \geq 1} (1 + xq^i) = \sum_{n \geq 0} \frac{x^n q^{n+1}}{(1 - q)(1 - q^2) \cdots (1 - q^n)}.
\]

(b) Show that the two identities are special cases of Cauchy’s identity, above, and that they are limiting cases of the $q$-binomial theorem.

3. (a) Prove that the number of partitions of $n$ with no parts divisible by $d$ is equal to the number of partitions of $n$ with no part repeated $d$ or more times, for all $n$ and $d$.

(b) Prove that the number of partitions of $n$ in which each part $j$ is repeated less than $j$ times is equal to the number of partitions of $n$ in which no part is a square.

4. Let $p_+(n)$ be the the number of partitions of $n$ with an even number of parts and $p_-(n)$ the number with an odd number of parts. Let $p_{DO}(n)$ be the number of partitions of
n with distinct odd parts, and let $k(n)$ be the number of partitions of $n$ which are conjugate to themselves. Prove that

$$k(n) = p_{DO}(n) = (-1)^n(p_+(n) - p_-(n))$$

**Lecture 4**

Homework for Lectures 3 and 4 due Feb. 7.

1. From the $q$-binomial theorem

$$
\prod_{i=0}^{m-1} (1 + xq^i) = \sum_{j=0}^{m} \binom{m}{j}_q q^{\binom{j}{2}} x^j,
$$

deduce

$$
\prod_{i=1}^{s} (1 + x^{-1}q^i) \prod_{i=0}^{t-1} (1 + xq^i) = \sum_{j=-s}^{t} \binom{s+t}{s+j}_q q^{\binom{j}{2}} x^j.
$$

By letting $s$ and $t$ go to infinity, prove *Jacobi’s triple product identity*:

$$
\sum_{j \in \mathbb{Z}} (-1)^j a^{\binom{j}{2}} x^j = \prod_{i \geq 1} \frac{1}{1 - xq^i}.
$$

2. Deduce Euler’s pentagonal number theorem by letting $a$ and $x$ go to suitable powers of $q$ in Jacobi’s triple product identity.

3. Let $t(\lambda)$ denote the number of different parts of $\lambda$ and $m(\lambda)$ the smallest nonzero part (when $\lambda$ is non-empty).

(a) Prove that

$$
1 + \sum_{\lambda \neq \emptyset} (-1)^{m(\lambda)-1} \frac{t(\lambda) - 1}{m(\lambda) - 1} q^{\binom{m(\lambda)}{2}} x^{t(\lambda)} = \sum_{n \geq 0} (-1)^n x^n q^{\binom{3n^2+n}{2}} \prod_{i=1}^{n} \frac{1}{1 - q^i} \prod_{i=n+1}^{\infty} \frac{1}{1 - xq^i},
$$

by counting pairs $(\lambda, T)$ with sign $(-1)^{|T|}$, where $T$ is a subset of the set of different parts of $\lambda$ and $|T| < m(\lambda)$.

(b) By changing the condition on $(\lambda, T)$ to require that $|T| = m(\lambda)$ and $m(\lambda) \in T$, prove that

$$
\sum_{\lambda \neq \emptyset} (-1)^{m(\lambda)} \frac{t(\lambda) - 1}{m(\lambda) - 1} q^{\binom{m(\lambda)}{2}} x^{t(\lambda)} = \sum_{n \geq 1} (-1)^n x^n q^{\binom{3n^2-n-1}{2}} \prod_{i=1}^{n-1} \frac{1}{1 - q^i} \prod_{i=n}^{\infty} \frac{1}{1 - xq^i}.
$$

(c) Deduce *Sylvester’s identity*

$$
\prod_{i \geq 1} (1 - xq^i) = 1 + \sum_{n \geq 1} (-1)^n x^n \left( q^{\binom{3n^2+n}{2}} \prod_{i=1}^{n} \frac{1 - xq^i}{1 - q^i} + q^{\binom{3n^2-n}{2}} \prod_{i=1}^{n-1} \frac{1 - xq^i}{1 - q^i} \right),
$$
which is another generalization of Euler’s pentagonal number theorem.

4. The following two identities are due to Gauss:

$$\sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} = \prod_{i \geq 1} \frac{1 - q^i}{1 + q^i};$$

$$\sum_{n \geq 0} q^{\binom{n+1}{2}} = \prod_{i \geq 1} \frac{1 - q^{2i}}{1 - q^{2i-1}}.$$

(a) Interpret them combinatorially as partition identities.
(b) Prove them, either combinatorially (not so easy) or using Jacobi’s triple product identity.

5. The Durfee square of a partition $\lambda$ is the largest $k \times k$ square that fits inside its Young diagram.

(a) Use Durfee squares to prove the identities in Stanley, Ch. 1, Ex. 23(b,d).
(b) Use Durfee squares for partitions with distinct parts to derive an identity analogous to the one in Stanley, Ex. 23(b), and from it obtain another proof of Sylvester’s identity.

Lecture 5

1. Find a Pascal triangle-like recurrence for the signless Stirling numbers $c(n, k)$ of the first kind (= number of permutations $\pi \in S_n$ with $k$ cycles), and use it to compute a table for $n \leq 5$.

2. The $q$-Stirling numbers are defined by the recurrence

$$S_q(n, k) = (k)_q S_q(n - 1, k) + q^{k-1} S_q(n - 1, k - 1),$$

with initial condition $S_q(0, k) = \delta_{0,k}$. Prove the following $q$-analog of the classical formula

$$x^n = \sum_k S(n, k)(x)_n = ( (r)_q^n = \sum_k S_q(n, k)(r)_q(r-1)_q \cdots (r-n+1)_q.$$

3. Show that the $q$-Stirling numbers defined in the previous problem have the ordinary generating function (for each $k$)

$$\sum_n S_q(n, k)x^n = \frac{q^k(1)_x}{(1-x)(1-(2)_q)x \cdots (1-(k)_qx)}.$$

Recall from the lecture (see also Stanley, Ch. 1, Ex. 16) that there is a simple bijection between partitions of $1, \ldots, n$ with $k$ blocks and words $w$ of the type described in Lecture 1, Problem 8. Show that $S_q(n, k) = \sum w q^{|w|}$, where the sum is over these words $w$, and $|w| = \sum_{i=1}^n (w_i - 1)$.

Lecture 6
1. Two species $F$ and $G$ are equivalent if they are naturally isomorphic as functors. More precisely, this means that one can give for each finite set $S$ a bijection $\psi_S : F(S) \to G(S)$ such that the triple $(F(S), G(S), \psi_S)$ is functorial in $S$, in the category whose objects are triples consisting of two finite sets and and arrow between them, and whose arrows between two such triples are commutative squares with the given triples as left and right sides.

(a) If $L$ is the species of linear orderings, and $P$ is the species of permutations, show that $L$ and $P$ are not equivalent.

(b) The Hadamard product $F \ast G$ of two species is the species $(F \ast G)(S) = F(S) \times G(S)$. Show that the species $L \ast L$ and $P \ast L$ are equivalent. This can be understood as an explanation of the fact that the inequivalent species $L$ and $P$ have the same exponential generating function.

2. An unordered binary tree is a rooted tree in which each non-leaf node has two children, but we do not order the children. Find the exponential generating function which counts unordered binary trees on $n$ labelled nodes.

3. An at most binary tree is an unordered rooted tree in which each node has at most two children. Find the exponential generating function which counts at most binary trees on $n$ labelled nodes.

Lecture 7

Homework for Lectures 5-7 due Feb. 14.

In the following problems I use the ‘double factorial’ notation $n!!$ for the product $(2n - 1)(2n - 3) \cdots 3 \cdot 1$ of the first $n$ odd numbers.

1. A perfect matching on a set $S$ of $2n$ elements is a partition of $S$ into $n$ blocks of two elements each. Perfect matchings form a species $M$, with $M(S) = \emptyset$ if $|S|$ is odd.

(a) Find the exponential generating function for the species of perfect matchings.

(b) Deduce algebraically that the number of perfect matchings on a set of $2n$ elements is $n!!$.

(c) Give a direct counting argument for the result in (b).

2. An involution of $S$ is a permutation $\tau$ of $S$ such that $\tau^2 = 1$.

(a) Find the exponential generating function $\sum_n t_n x^n / n!$, where $t_n$ is the number of involutions of an $n$ element set.

(b) Show that $t_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} k!!$.

3. Let $e(2n)$ be the number of permutations $\sigma$ of a set of $2n$ elements with the property that every cycle of $\sigma$ has even length.

(a) Find the exponential generating function $\sum_n e(2n)x^{2n}/(2n)!$.

(b) Deduce that $e(2n) = (n!!)^2$.

4. (a) Problems 1(b) and 3(b) show that the number of permutations of a $2n$ element set $S$ with only even cycles is equal to the number of pairs of perfect matchings on $S$. Construct a direct bijection between the two (to do this you will probably need to fix the set $S$ to be $[2n]$ and use the numerical values of its elements to make some auxiliary choices).
(b) Show that the species of permutations with even-length cycles and the species of pairs of perfect matchings are not equivalent. This can be understood as explaining the need for auxiliary choices in part (a).

5. (a) Find the exponential generating function \( \sum_{n} r_n x^n/n! \), where \( r_n \) is the number of permutations of odd order of an \( n \) element set.

(b) Deduce that \( r_{2n} = (n!!)^2 \) and \( r_{2n+1} = (2n + 1)(n!!)^2 \).

6. Let \( g(n, k) \) denote the number of connected simple graphs (i.e., without loops or multiple edges) on \( n \) labelled vertices with \( k \) edges. Derive the mixed ordinary/exponential generating function

\[
\sum_{n=1}^{\infty} \sum_{k} g(n, k)q^k x^n/n! = \log \sum_{n=0}^{\infty} (1 + q)^{\binom{n}{2}} x^n/n!
\]

and use it to compute \( \sum_k g(n, k)q^k \) for all \( n \leq 4 \). As a check, count the graphs in question by hand and compare answers.

Lecture 8

1. In the lecture we showed that the number of unlabelled ordered trees with \( n + 1 \) vertices is equal to the Catal\'an number \( C_n = \frac{(2n)!}{n!(n+1)} \).

(a) Show that the number of binary trees with \( 2n + 1 \) vertices (every binary tree has an odd number of vertices) is also equal to \( C_n \).

(b) A lisp tree is an at most binary tree in which the children of each node are distinguished as left and right. That is, if there are two children, they are ordered, and if there is one child, we still distinguish two cases. Show that the number of lisp trees with \( n \) vertices is equal to \( C_n \).

(c) Find bijections between the above mentioned three kinds of trees enumerated by \( C_n \).

2. (a) The diameter \( d \) of a tree \( T \) is the maximum length of a path in \( T \) (a path with \( n \) vertices has length \( n - 1 \)). Prove that if \( d \) is even then all paths of length \( d \) have the same middle vertex, called the center of \( T \), and if \( d \) is odd, then all paths of length \( d \) have the same middle edge; the path of length 1 consisting of this edge and its two endpoints is called the bicenter of \( T \).

(b) Show that if \( d \) is even, the species of labelled unrooted trees of diameter \( d \) is equivalent to the species of labelled rooted trees of height \( d/2 \) with the property that at least two children of the root are roots of subtrees of height \( d/2 - 1 \).

(c) Show that if \( d \) is odd, the species of labelled unrooted trees of diameter \( d \) is equivalent to the species of unordered pairs of disjoint rooted trees of height \( (d - 1)/2 \).

(d) Let \( T_h \) be the species of labelled rooted trees of height \( h \), and let \( T_{\leq h} = T_0 + \cdots + T_h \). Show that these are given by the recurrence

\[
T_0 = X \\
T_h = X((E - 1) \circ T_{h-1})(E \circ T_{\leq h-2}) \quad \text{for } h > 0.
\]
(e) Using products and composition of species, express the species of labelled unrooted trees of diameter \(d\) in terms the species \(T_h\) for various \(h\).

(f) Use the above to calculate the number of labelled unrooted trees of diameter \(d\) on \(n\) vertices, for \(n \leq 5\) and all \(d\). Check your answer by summing over \(d\).

Lecture 9

1. Deduce the result of Lecture 8, Problem 1(a) from Cayley’s formula.

2. In the lecture we proved the following version of Cayley’s formula:

\[
\sum T \prod_j x^{c_T(j)}_j = (x_1 + \cdots + x_n)^{n-1},
\]

where \(T\) ranges over rooted trees on \([n]\), and \(c_T(j)\) is the number of children of vertex \(j\) in \(T\).

(a) Prove the following variant of Cayley’s formula:

\[
\sum T \prod_j x^{d_T(j)}_j = x_1 \cdots x_n (x_1 + \cdots + x_n)^{n-2},
\]

where \(T\) ranges over unrooted trees on \([n]\), and \(d_T(j)\) is the degree (i.e., number of neighboring vertices) of vertex \(j\) in \(T\).

(b) Show that (a) implies that the sum \(\sum_T \prod_j x^{c_T(j)}_j\) over trees with root \(j\) is equal to \(x_j(x_1 + \cdots + x_n)^{n-2}\), which implies the version of Cayley’s formula we proved in the lecture.

3. A rooted forest is a disjoint union of rooted trees Prove the identity

\[
\sum F \prod_j x^{c_F(j)}_j = \binom{n-1}{k-1} (x_1 + \cdots + x_n)^{n-k},
\]

where \(F\) ranges over rooted forests with \(k\) components on vertex set set \([n]\). In particular, the number of such forests is \(\binom{n-1}{k-1} n^{n-k}\).

4. (a) Show that the exponential generating function for the species of labelled unrooted forests is given by

\[
F(x) = \exp \sum_{n=1}^{\infty} \frac{n^{n-2} x^n}{n!}.
\]

(b) Use (a) to calculate the number of labelled unrooted forests on \([n]\) for \(n \leq 6\).

(c) Modify (a) to get a mixed ordinary/exponential generating function \(F(t, x)\) for the species of labelled unrooted forests, in which the coefficient of \(t^k\) forest with \(k\) components. Use this to refine your answer to (b) to count forests by number of components.

5. A leaf-labelled tree \(T\) is a rooted tree whose leaves are the elements of a given set \(S\), and whose other nodes are unlabelled. Let us require each non-leaf node of \(T\) to have at least two children; then there are finitely many leaf-labelled trees on a given finite leaf set \(S\).
Introduce indeterminates \(a_2, a_3, \ldots\) and assign each leaf-labelled tree the weight \(w(T) = \prod_i a_i^{r_i}\), where \(r_i\) is the number of non-leaf nodes in \(T\) with \(i\) children. Then show that the mixed ordinary/exponential generating function enumerating leaf-labelled trees with these weights is the functional composition inverse of \(x - A(x)\), where \(A(x) = \sum_{n \geq 2} a_n x^n / n!\).

6. Recall the notation used in the lecture: given a formal series \(E(x) = e_0 + e_1 x + e_2 x^2 + \cdots\), where \(e_0\) is invertible, let \(F(x)\) be the functional composition inverse of \(G(x) = x/E(x)\), i.e., \(F(x)\) satisfies the functional equation

\[
F(x) = xE(F(x)).
\]

We saw that the coefficient \([x^n]F(x)\) is the ordinary generating function enumerating (unlabelled) ordered rooted trees (Stanley calls them plane trees) on \(n\) vertices, with weight \(\prod v \in T e_d(v)\), where \(d(v)\) is the number of children of vertex \(v\) in \(T\). In class we gave a combinatorial proof of the case \(k = 1\) of the following more general form of the Lagrange inversion formula:

\[
[x^{n+k}] F(x)^k = \frac{k}{n+k} [x^n] E(x)^{n+k}.
\]  

(a) Show that for any sequence \((r_1, \ldots, r_{n+k}) \in \mathbb{N}^{n+k}\) such that \(r_1 + \cdots + r_{n+k} = n\), exactly \(k\) of its rotations \((s_1, \ldots, s_{n+k}) = (r_{l+1}, \ldots, r_{n+k}, r_1, \ldots, r_l)\) satisfy

\[
s_1 + \cdots + s_i + k > i \quad \text{for all} \quad 0 \leq i < n + k.
\]

(b) Show that

\[
\frac{k}{n+k} \sum_{r_1 + \cdots + r_{n+k} = n} e_{r_1} \cdots e_{r_{n+k}} = \sum_{\lambda \subseteq (n+k-2, \ldots, k-1)} \prod_{i=0}^{n+k-1} e_{\alpha_i(\lambda)},
\]

where \(\lambda\) ranges over partitions whose Young diagram fits inside that of \((n+k-2, \ldots, k-1)\), and \(\alpha_i(\lambda)\) is the number of parts equal to \(i\) in \(\lambda\), with \(\alpha_0(\lambda)\) defined so that \(\sum_i \alpha_i(\lambda) = n\).

(c) Use (b) to give a combinatorial proof of the generalized Lagrange inversion formula above.

7. Derive a formula for the number of unlabelled ordered rooted forests of \(k\) trees on \(n\) vertices.

8. Prove that the number of ordered rooted trees with \(n+1\) vertices and \(j\) leaves is equal to

\[
\frac{1}{n+1} \binom{n+1}{j} \binom{n-1}{n-j}.
\]

9. Prove that the number of ways to subdivide an \(n\)-gon into \(k\) polygons by introducing \(k-1\) diagonals that do not intersect except at their endpoints is equal to the number of
ordered rooted trees with \( n + k - 1 \) vertices, \( n - 1 \) leaves, and no vertices with exactly one child. Derive the formula

\[
\frac{1}{n + k - 1} \binom{n + k - 1}{n - 1} \binom{n - 3}{n - k - 2}
\]

for this number. In particular, taking \( k = n - 2 \), deduce that the number of triangulations of an \( n \)-gon is the Catalan number \( C_{n-2} \). In this problem the \( n \)-gon is regarded as fixed in place, so for example the two triangulations of a square count as different even though they are the same up to symmetry. See Stanley, Exercise 6.19 for 65 more combinatorial interpretations of Catalan numbers.

**Lecture 10**

1. (a) Verify by direct calculation that the cycle index \( Z_C \) for the species of cyclic orderings (\( i.e. \), permutations with one cycle) is given by

\[
Z_C = \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \frac{1}{1 - p_n},
\]

where \( \phi \) is Euler’s totient function: \( \phi(n) \) is the number of integers \( r \in [n] \) relatively prime to \( n \).

(b) Check that \( Z_C(x, 0, \ldots) \) and \( Z_C(x, x^2, \ldots) \) agree, respectively, with the exponential generating function for the species of cyclic orderings, and the ordinary generating function for cyclic orderings up to isomorphism.

2. Recall that the cycle index of the trivial species is given by

\[
Z_E = \exp \sum_{n=1}^{\infty} \frac{p_n}{n}.
\]

Verify that the plethysm \( Z_E \ast Z_C \) agrees with the formula we obtained by direct calculation for the cycle index of the species of permutations,

\[
Z_P = \prod_{n=1}^{\infty} \frac{1}{1 - p_n}.
\]

**Lecture 11**

1. (a) Find an explicit formula for the cycle index \( Z_I \) of the species of involutions, \( I(S) = \{ \text{involutions } \sigma: S \to S \} \).

(b) Evaluate \( Z_I[x] \) and verify that it agrees with the obvious ordinary generating function counting involutions up to conjugacy.

2. Stanley, Exercise 5.4(a).

3. (a) Stanley, Exercise 5.5. A proper \( n \)-coloring of a graph \( G = (V, E) \) is a function \( c : V \to [n] \) such that \( c(v) \neq c(w) \) whenever \( v, w \) are the endpoints of an edge \( e \in E \). A graph \( G \) is *bipartite* if there exists a proper 2-coloring of \( G \).
(b) Let $B$ be the species $B(S) = \{\text{bipartite graphs with vertex set } S\}$ and let $G_2$ be the species $G_2(S) = \{\text{graphs } G \text{ with vertex set } S, \text{ together with a proper 2-coloring of } G\}$. Part (a) implies that $G_2$ and $B^2$ have the same exponential generating function, and indeed the same mixed generating function weighted by number of edges. Are the species $B^2$ and $G_2$ equivalent?

4. Stanley, Exercise 5.9.
5. Stanley, Exercise 5.10(a,c).
6. Stanley, Exercises 5.11.

Lecture 12

1. Use the results of Lecture 8, problem 2, to find a recurrence giving the ordinary generating functions $Z_{T_h}(x)$ for unlabelled rooted trees of height $h$, and an expression in terms of these for the ordinary generating functions $Z_{U_d}(x)$ for unlabelled unrooted trees of diameter $d$. From this, calculate the number of unlabelled unrooted trees of diameter $d$ on $n$ vertices for $n \leq 5$ and all $d$.

2. Use species generating function methods to find the number of unlabelled unrooted forests on $n$ vertices for $n \leq 6$.

Lecture 13

Homework for Lectures 8-13 due Feb. 28.

1. In this problem we work out an alternate proof of the Matrix–Tree theorem,

$$\det M_n = \sum_{F} \left( \prod_{\text{roots } i} z_i \right) \left( \prod_{\text{edges } (i,j)} x_{ij} \right),$$

where $M_n$ is $n \times n$ with off-diagonal entries $-x_{ij}$ and diagonal entries $z_i + \sum_{j \neq i} x_{ij}$.

(a) Setting $z_i = x_{ii}$, show that every monomial appearing in $\det M_n$ has the form $m_f(x) = \prod_{i=1}^{n} x_{i,f(i)}$ for some function $f : [n] \to [n]$.

(b) Evaluate the coefficient of $m_f(x)$ by setting $x_{ij} = 1$ for $j = f(i)$ and $x_{ij} = 0$ for $j \neq f(i)$. Show that the result is equal to zero if $f$ has any cycle $v,f(v),f(f(v)),\ldots,f^{(k)}(v) = v$ of length $k > 1$, and otherwise equal to one. Hint: by symmetry among the vertex labels, you can assume each cycle has the form $i, f(i) = i + 1, f(i + 1) = i + 2, \ldots, f(i + k - 1) = i$, and that $f(j) > j$ for all other $j$. The matrix then becomes block-triangular.

2. Let $f_m(r)$ be the number of rooted spanning forests with $r$ roots in the graph $C_m$, a cycle on $m$ vertices ($m > 1$). Prove that

$$F_m(z) \overset{\text{def}}{=} \sum_r f_m(r) z^r = \prod_{j=0}^{m-1} (z + 2 - 2 \cos(2\pi j/m)) = \sum_{r=1}^{m} \frac{m}{r} \binom{m + r - 1}{2r - 1} z^r.$$

3. The product $G \times H$ of two simple graphs (graphs without loops or multiple edges) is the graph on vertex set $V(G) \times V(H)$ with edges $\{(v,w),(v',w')\}$ for $v = v'$ and $\{w,w'\} \in E(H)$
or $w = w'$ and $\{v, v'\} \in E(G)$. The adjacency matrix $A_G$ of a graph $G$ on $n$ vertices is the $n \times n$ matrix with rows and columns labelled by the vertices, and entries $(A_G)_{v,w} = 1$ if $\{v, w\} \in E(G)$, zero otherwise. Let $D_G$ be the diagonal matrix whose $(v,v)$ entry is the degree of $v$.

(a) Let $f_G(r)$ be the number of rooted spanning forests of $G$ with $r$ roots, and let $F_G(z) = \sum_r f_G(r)z^r$ be the corresponding generating function. Show that $F_G(z) = \prod_i (z + \alpha_i)$, where the $\alpha_i$’s are the eigenvalues of $D_G - A_G$.

(b) Show that $F_G \times H(z) = \prod_{i,j} (z + \alpha_i + \beta_j)$, where $F_G(z) = \prod_i (z + \alpha_i)$ and $F_H(z) = \prod_j (z + \beta_j)$. In particular, the numbers $f_G(r)$ and $f_H(r)$ for all $r$ determine the corresponding numbers $f_{G \times H}(r)$.

(c) Show that if $Q_n$ is the graph formed by the vertices and edges of the $n$-cube, that is, the product of $n$ copies of the complete graph on 2 vertices, then

$$F_{Q_n}(z) = \prod_{k=0}^{n} (z + 2k)^{\binom{n}{k}}.$$ This generalizes Stanley, Exercise 5.6.10, which follows by taking the coefficient of $z$.

4. Let $G$ be an undirected simple graph with vertex set $[n]$ and all vertex degrees $d_i$ even. Let $M_G$ be the square matrix with off-diagonal entries

$$\frac{x_j}{x_i + x_j} \quad \text{if} \quad \{i, j\} \text{ is an edge of } G,
0 \quad \text{otherwise},$$

and all row-sums zero. Show that the number of closed Eulerian walks in $G$ is given by the coefficient of $\lambda x_1^{d_1/2} \cdots x_n^{d_n/2}$ in

$$\frac{|E(G)|}{n} \prod_i (d_i/2 - 1)! \prod_{\{i,j\} \in E(G)} (x_i + x_j) \det(M_G + \lambda I).$$

Lectures 14-23

1. A plane partition of $n$ is a sequence of ordinary partitions $\lambda = (\lambda^{(1)} \supseteq \cdots \supseteq \lambda^{(k)})$ of total size $\sum_i |\lambda^{(i)}| = n$, weakly decreasing in the sense that the diagram of each $\lambda^{(i)}$ is contained in that of the preceding one. The diagram of a plane partition is the three-dimensional array in $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ whose $i$-th horizontal layer is the diagram of $\lambda^{(i)}$.

Find a bijection between plane partitions whose diagram fits inside a $k \times l \times m$ box and semistandard Young tableaux of shape $(k^l)$ with entries in $[l + m]$.

2. Prove the formula for complete homogeneous symmetric functions in terms of elementary symmetric functions

$$h_n = \sum_{|\lambda| = n} (-1)^{n-l(\lambda)} \binom{l(\lambda)}{r_1, r_2, \ldots, r_k} e_{\lambda},$$
where \( \lambda = (1^{r_1}, 2^{r_2}, \ldots, k^{r_k}) \).

3. Show that the dominance partial order on partitions of \( n \) satisfies
\[
\lambda \leq \mu \iff \lambda' \geq \mu',
\]
where prime denotes the transpose partition.

4. For \( 1 \leq i < j \leq n \), define the raising operator \( R_{ij} \) on \( \mathbb{Z}^n \) by
\[
R_{ij}(\nu_1, \ldots, \nu_n) = (\nu_1, \ldots, \nu_i + 1, \ldots, \nu_j - 1, \ldots, \nu_n).
\]
(a) Show that the dominance order \( \leq \) is the transitive closure of the relation on partitions \( \lambda \rightarrow \mu \) if \( \mu = R_{ij} \lambda \) for some \( i < j \).

(b) We say that \( \mu \) covers \( \lambda \) if \( \lambda < \mu \) and there is no \( \nu \) such that \( \lambda < \nu < \mu \). Show that \( \mu \) covers \( \lambda \) if and only if \( \mu = R_{ij} \lambda \), where \( i, j \) satisfy the following condition: either \( j = i + 1 \), or \( \lambda_i = \lambda_j \) (or both).

(c) Find the smallest \( n \) such that the dominance order on partitions of \( n \) is not a total ordering, and draw its Hasse diagram (i.e., the graph of the covering relation).

5. (a) Use the fundamental theorem of symmetric functions to show that if \( f(t) = t^k + a_1 t^{k-1} + \cdots + a_k \) and \( g(t) = t^l + b_1 t^{l-1} + \cdots + b_l \), there is a polynomial \( R_{f,g}(a_1, \ldots, a_k, b_1, \ldots, b_l) \) such that \( R_{f,g} = 0 \) if and only if \( f \) and \( g \) have a common root. The minimal such polynomial (which is unique up to a constant factor) is called the resultant of \( f \) and \( g \). Calculate \( R_{f,g} \) for \( k = 2 \) and \( l = 3 \).

(b) Show that \( R_{f,g} \) is the determinant of the \((k + l) \times (k + l)\) matrix
\[
\begin{bmatrix}
1 & a_1 & \ldots & a_l & 0 & \ldots & 0 & 0 \\
0 & 1 & a_1 & \ldots & a_l & 0 & \ldots & 0 \\
\vdots & & & & & & & \\
0 & 0 & \ldots & 0 & 1 & a_1 & \ldots & a_l \\
1 & b_1 & \ldots & b_k & 0 & \ldots & 0 & 0 \\
0 & 1 & b_1 & \ldots & b_k & 0 & \ldots & 0 \\
\vdots & & & & & & & \\
0 & 0 & \ldots & 0 & 1 & b_1 & \ldots & b_k
\end{bmatrix}
\]

6. Express \( m_\lambda(1, 1, \ldots, 1) \), with \( n \) ones, as a more familiar combinatorial quantity.

7. The symmetric functions \( f_\lambda = \omega m_\lambda \) are sometimes called the “forgotten” symmetric functions. Show that the matrix of coefficients of the forgotten symmetric functions \( f_\lambda \) expressed in terms of monomial symmetric functions \( m_\lambda \) is the transpose of the matrix of the elementary symmetric functions \( e_\lambda \) expressed in terms of the complete homogeneous symmetric functions \( h_\lambda \).

8. For any symmetric polynomial \( f \), let \( f^\perp \) be the operator adjoint to multiplication by \( f \) with respect to the Hall inner product, that is, \( \langle f^\perp g, h \rangle = \langle g, fh \rangle \) for all \( g, h \).
(a) Find a formula for \( h_k m_\lambda \), expressed again in terms of monomial symmetric functions \( m_\mu \).

(b) Show that the basis of monomial symmetric functions is uniquely characterized by the formula from part (a).

9. Let \( \partial p_k \) be the operator on symmetric functions given by partial differentiation with respect to \( p_k \), under the identification of symmetric functions with polynomials \( f \in \mathbb{Q}[p_1, p_2, \ldots] \). Show that \( \partial p_k \) is adjoint with respect to the Hall inner product to the operator of multiplication by \( p_k/k \).

10. (a) Show that the coefficient of \( m_\lambda[X] m_\nu[Y] \) in \( m_\nu[X + Y] \) is equal to 1 if \( \nu = \lambda \cup \mu \), and zero otherwise.

(b) Use part (a) and the fact that \( \langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} \) to show that the plethystic substitution \( X \rightarrow X + Y \) is adjoint to multiplication, in the sense that for all symmetric polynomials \( f \), \( g \), \( h \) we have

\[
\langle f, gh \rangle = \langle f[X + Y], g[X]h[Y] \rangle_{XY},
\]

where \( \langle -, - \rangle_{XY} \) is the inner product on \( \Lambda(X) \otimes \Lambda(Y) \) defined so that \( \{ m_\lambda[X] m_\nu[Y] \} \) and \( \{ h_\lambda[X] h_\nu[Y] \} \) are dual bases.

11. If \( A \) is an algebra over a field \( k \), then \( A \otimes_k A \) is an algebra with multiplication characterized uniquely by \( (a \otimes b)(c \otimes d) = ac \otimes bd \). A coproduct is an algebra homomorphism \( \Delta: A \rightarrow A \otimes A \) which is coassociative in the sense that the two maps \((1 \otimes \Delta) \circ \Delta\) and \((\Delta \otimes 1) \circ \Delta\) from \( A \) to \( A \otimes A \otimes A \) are equal. This axiom is dual to the associative law for multiplication \( \mu: A \otimes A \rightarrow A \), which can be formulated as \( \mu \circ (1 \otimes \mu) = \mu \circ (\mu \otimes 1) \). Here \( 1: A \rightarrow A \) denotes the identity map.

Taking \( \Lambda \) to be the algebra of symmetric functions with coefficients in \( k = \mathbb{Q} \), and identifying \( \Lambda \otimes \Lambda \) with \( \Lambda(X) \Lambda(Y) \), show that \( \Delta(f) = f[X + Y] \) defines a coproduct on \( \Lambda \).

Remark: An algebra equipped with a coproduct is called a bialgebra. If we also define the counit \( \epsilon: \Lambda \rightarrow \mathbb{Q} \) by \( \epsilon(f) = \langle f, 1 \rangle = f[0] \) and the antipode \( S: \Lambda \rightarrow \Lambda \) by \( Sf = f[-X] \), these together with \( \Delta \) can be shown to satisfy the axioms of a Hopf algebra.

12. Consider an alphabet \( \mathcal{A} = \{ x_1, y_1, x_2, y_2, \ldots \} \) of two kinds of variables. Fix any ordering of \( \mathcal{A} \) and define the “super” Schur function \( s_\lambda(x; y) \) to be the generating function for “super” semistandard Young tableau of shape \( \lambda \). Such a tableau is a filling of the diagram of \( \lambda \) by variables from \( \mathcal{A} \), weakly increasing along rows and columns as usual, with the requirement that no \( x_i \) is repeated in any column and no \( y_i \) is repeated in any row. In particular, \( s_\lambda(x; 0) = s_\lambda(x) \) and \( s_\lambda(0; y) = s_\lambda(y) \).

(a) Prove that \( s_\lambda(x; y) \) is symmetric in the \( x_i \)'s and the \( y_i \)'s separately, and does not depend on the ordering chosen for \( \mathcal{A} \).

(b) Prove that if \( y_1 = x_1 \), then \( s_\lambda(x_1, x_2, \ldots; -y_1, -y_2, \ldots) = s_\lambda(x_2, \ldots; -y_2, \ldots) \).

(c) Prove that, in plethystic notation,

\[
s_\lambda(x; -y) = s_\lambda[X - Y].
\]
(d) Prove that
\[ s_{(n^m)}(x_1, \ldots, x_n; -y_1, \ldots, -y_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x_i - y_j). \]

(e) Prove that the resultant \( R_{f,g} \) from Problem 5 is given by the expansion of \( s_{(\mu')}[X - Y] \) in terms of elementary symmetric functions \( e_j[X] \) and \( e_j[Y] \), where these are equated with the coefficients of \( f \) and \( g \) by the rule \( a_j = (-1)^j e_j[X] \), \( b_j = (-1)^j e_j[Y] \).

13. Let \( \epsilon \) be a fictitious alphabet such that \( p_k[\epsilon] = \delta_{1,k} \). Stated more correctly, this means we are to interpret \( f[\epsilon] \) as the image of \( f \) under the homomorphism \( \Lambda \to \mathbb{Q} \) mapping \( p_k \) to \( \delta_{1,k} \).

(a) Prove the identity \( f[\epsilon] = (f, \exp(p_1)). \)
(b) Prove the identity \( f[\epsilon] = \lim_{n \to \infty} (f[nx])_{x \to 1/n}. \)
(c) Show that \( e_k[\epsilon] = h_k[\epsilon] = 1/n! \).
(d) More generally, show that \( s_\lambda[\epsilon] = f_\lambda/n! \), where \( |\lambda| = n \) and \( f_\lambda \) is the number of standard Young tableaux of shape \( \lambda \).

14. (a) Recall from class that \( h_n = \sum_{|\lambda|=n} p_{\lambda}/z_\lambda \), where \( z_\lambda = \prod_i i^{r_i} r_i! \) for \( \lambda = (1^{r_1}, 2^{r_2}, \ldots) \). Show that this is equivalent to Newton’s determinant formula
\[
 h_n = \frac{1}{n!} \det \begin{bmatrix}
 p_1 & -1 & 0 & \ldots & 0 \\
 p_2 & p_1 & -2 & \ldots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 p_{n-1} & p_{n-2} & \ldots & -(n-1) & 0 \\
 p_n & p_{n-1} & \ldots & 1 & 1
\end{bmatrix}
\]

(b) Show that \( e_n \) is given by the same determinant without the minus signs.

15. Prove the identity \( s_{(n-1,n-2,\ldots,1)}(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i + x_j) \).

[The previous two problems and Problem 1 for Lectures 24-26 are from Macdonald’s book Symmetric Functions and Hall Polynomials.]

16. Prove that standard tableaux \( S \) and \( T \) are dual equivalent if and only if there exist standard tableaux \( S' \) and \( T' \) of some normal shape \( \lambda \) and a tableau \( X \) (of a shape \( \nu \) for which \( \lambda \sqcup \nu \) makes sense) such that \( J^X(S') = S \) and \( J^X(T') = T \).

17. (a) Let \( S \) and \( T \) be standard tableau of the same (skew) shape \( \nu \), where \( |\nu| = 3 \). Verify directly that if there exists a slide into the same cell that gives tableaux of different shapes when applied to \( S \) and \( T \), then the reading words of \( S \) and \( T \) must differ by a switch of entries in adjacent positions in the word. In particular, they cannot be \( \{213, 312\} \) or \( \{132, 231\} \).

(b) Use part (a) and the fact that jeu-de-taquin preserves descent sets to show that if \( S \) and \( T \) have reading words \( \{213, 312\} \) or \( \{132, 231\} \), then any slide into the same cell applied to \( S \) and \( T \) yields another pair of tableaux \( S', T' \) with the same shape and reading words \( \{213, 312\} \) or \( \{132, 231\} \). Deduce that any such pair is dual equivalent. [This avoids the case checking needed for the proof indicated in the lecture.]
18. Let $P$ be a partially ordered set with a least element $0$. Assume $P$ locally finite, which means that every interval $[0, x]$ is finite. Define a shape to be a finite subset $\nu \subseteq P$ such that $x \leq y \leq z$ and $x, z \in \nu$ imply $y \in \nu$; define a standard tableau of shape $\nu$ to be an order-preserving bijection $\nu \to [n]$, where $n = |\nu|$. If $P$ is $\mathbb{N} \times \mathbb{N}$, these reduce to the usual notions of (skew) shapes and tableaux. Define forward jeu-de-taquin slides for tableaux on $P$ analogously to the definition for $P = \mathbb{N} \times \mathbb{N}$.

One says that $P$ has the jeu-de-taquin property if for every tableau $T$ and every sequence of forward slides that carries $T$ into a shape containing $0$, the resulting tableau $S$ depends only on $T$ and not on the sequence of slides chosen. The fundamental theorem of jeu-de-taquin states that $P = \mathbb{N} \times \mathbb{N}$ has the jeu-de-taquin property. Show that the following posets have the jeu-de-taquin property.

(a) Trees (this is easy).

(b) The subset $R_k \subseteq \mathbb{Z} \times \mathbb{Z}$ which is the union of $\mathbb{N} \times \mathbb{N}$ and the set \{(1, −1), (0, −1), (0, −2), ..., (0, −k)\}.

(c) The subset $Q = \{(i, j) : i \leq j\} \subseteq \mathbb{N} \times \mathbb{N}$. This poset is called the shifted plane. Hint: set up a theory of dual equivalence for tableaux on $Q$ in which the elementary dual equivalences involve tableaux of size 4. You can find them all by starting with the unique shape of size 4 that contains 0 and possesses two distinct tableaux.

[All of the above are special cases of Proctor’s notion of $d$-complete posets (see J. Algebra 213 (1999), 272–303; J. Alg. Combinatorics 9 (1999), 61–94.).]

19. Consider the following sequence of operations on a tableau $T$ of straight shape $\lambda$:

1) delete the smallest entry;
2) perform a jeu-de-taquin slide into the now empty cell $(0, 0)$;
3) repeat steps (1) and (2) until all entries have been removed;
4) form the tableau $S$ whose entries $n, \ldots, 2, 1$ occupy the cells of $\lambda$ in the order they are vacated by step (2).

Show that the resulting tableau $S$ is equal to the evacuation $ev(T)$.

20. Let $\lambda = (k, 1^l)$ be a hook shape. Prove the following rule for computing $s_\lambda s_\mu$, which generalizes the Pieri rules for $s_{(k)} s_\mu$ and $s_{(1^k)} s_\mu$:

(i) $s_\nu$ occurs with non-zero coefficient in $s_\lambda s_\mu$ only if $\nu / \mu$ is a disjoint union of ribbons (connected skew shapes containing no $2 \times 2$ rectangle) of total size $k + l$,

(ii) in that case, the coefficient is $\binom{r-1}{h-l-1}$, where $\mu / \nu$ consists of $r$ ribbons, and $h$ is the sum of their heights (interpreting $\binom{r-1}{h-l-1}$ as zero if $h - l - 1$ is negative).

Lectures 24–26

Homework for Lectures 14-26 due April 11.

1. Let $|\lambda| = |\mu| = n$. Show that $\langle h_\lambda, h_\mu \rangle$ is equal to the number of double cosets $S_\lambda w S_\mu$ in the symmetric group $S_n$, where $S_\lambda$ and $S_\mu$ are Young subgroups of $S_n$. 

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2. Show that $G$ is abelian if and only if all the irreducible characters of $G$ are one-dimensional.

3. Use Maschke’s theorem to prove that if a square matrix $A$ over $\mathbb{C}$ satisfies $A^m = I$ for some $m$, then $A$ is diagonalizable. More generally, given several matrices $A_i$ such that $A_i^m = I$ and $A_iA_j = A_jA_i$ for all $i, j$, prove that they are simultaneously diagonalizable.

4. Find the character tables of
   (a) the dihedral group $D_8$ of order 8;
   (b) the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, where $i^2 = j^2 = k^2 = ijk = -1$, and signs multiply according to the usual rules.

   Show that $Q_8$ and $D_8$ are not isomorphic, and conclude that the character table of a finite group need not determine the group.

5. Let $V, W$ be representations of $G$. Show that if $V$ is irreducible and $W$ is one-dimensional, then $V \otimes W$ is irreducible.

6. Show that all the characters of $G$ are real if and only if $g$ is conjugate to $g^{-1}$ for all $g \in G$.

7. Let $k$ be the finite field with $p$ elements. The upper unit-triangular $2 \times 2$ matrices over $k$ form a matrix representation of $G = \mathbb{Z}/p\mathbb{Z}$. Show that the corresponding $G$-module $V$ is indecomposable but not irreducible, providing a counterexample to Maschke’s theorem over a field of prime characteristic.

8. Compute the character table of $S_5$.

9. If $V$ is a $G$-module and $H$ is a subgroup of $G$, then $H$ acts on $V$, so we can consider $V$ as an $H$-module, called the restriction of $V$ to $H$, and denoted $V|_H$. For this problem we will take $G = S_n$ and $H = A_n$, the alternating group. Denote by $V_\lambda$ the irreducible representation of $S_n$ whose character $\chi_\lambda$ corresponds to $s_\lambda$ via the Frobenius characteristic map.

   (a) Show that $V_\lambda|_{A_n} \cong V_{\lambda'}|_{A_n}$.

   (b) Show that $V_\lambda|_{A_n}$ is irreducible if $\lambda \neq \lambda'$, and that it is the direct sum of two inequivalent irreducible representations if $\lambda = \lambda'$. Also show that the irreducible constituents of $V_\lambda|_{A_n}$ are not isomorphic to those of $V_\mu|_{A_n}$ if $\{\lambda, \lambda'\} \neq \{\mu, \mu'\}$. [Hint: relate the character $\chi$ of $V_\lambda|_{A_n}$ to the $S_n$ character $\chi_\lambda + \chi_{\lambda'}$.]

   (c) Describe the restriction of the regular representation of $S_n$ to $A_n$. Deduce that every irreducible representation of $A_n$ occurs in the restriction of some irreducible representation of $S_n$.

   (d) Let $p(n)$ be the number of partitions of $n$, $p_{ee}(n)$ the number with an even number of even parts, and $k(n)$ the number of self-conjugate partitions, i.e., such that $\lambda = \lambda'$. Deduce from (a)–(c) above that
   
   \[ p_{ee}(n) = (p(n) + 3k(n))/2. \]

   (e) Recalling that $k(n)$ is equal to the number partitions of $n$ with distinct, odd parts, find a generating function proof of the identity in part (d).

10. Compute the character table of $A_5$. 

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11. (a) Let $V$ be a representation of $G$ with character $\chi$. Prove that $g \in G$ acts trivially on $V$ if and only if $\chi(g^k) = \chi(1)$ for all $k$.

(b) Show that every proper normal subgroup $H \subseteq G$ acts trivially on some non-trivial irreducible representation of $G$.

(c) Read off a proof that $A_5$ is simple from its character table.

12. Let $W$ be a $G$-module. Choose a complete set of mutually non-isomorphic irreducible representations $V_i$ of $G$ and set $T_i = \text{Hom}_G(V_i, W)$. Show that $W \cong \bigoplus_i V_i \otimes T_i$, where each vector space $T_i$ is regarded as a $G$-module with trivial $G$-action.