Math 249, Fall 2017
Problem Set 2

Due: Friday, Dec. 15

1. A *distribution* is a function together with a linear ordering on the preimage of each element of the codomain. Let J_k be the species such that $J_k(S)$ is the set of distributions from S to $\{1, \ldots, k\}$.

(a) Find the exponential generating function for J_k, and from it obtain the formula $(n+k-1)_n$ for the number of distributions from an n element set to a k element set.

(b) Do the same for the species of *surjective* distributions, obtaining the formula

$$\binom{n}{k}(n-1)_{n-k}$$

for the number of them.

2. Find the exponential generating function $D(x) = \sum_n D_n x^n/n!$, where D_n is the number of permutations $\sigma \in S_n$ with no fixed points. Deduce an explicit formula for D_n. [Permutations without fixed points are also called *derangements*. Compare Stanley, Example 2.2.1, where they are counted using the principle of inclusion and exclusion.]

3. A *perfect matching* on a set S of $2n$ elements is a partition of S into n blocks of two elements each. Perfect matchings form a species M, with $M(S) = \emptyset$ if $|S|$ is odd.

(a) Find the exponential generating function for the species of perfect matchings.

(b) Deduce algebraically that the number of perfect matchings on a set of $2n$ elements is $n!!$. Here and below the 'double factorial' notation $n!!$ stands for the product $(2n-1)(2n-3)\cdots 3\cdot 1$ of the first n odd numbers.

(c) Give a direct counting argument for the result in (b).

4. Let $e(2n)$ be the number of permutations σ of a set of $2n$ elements with the property that every cycle of σ has even length.

(a) Find the exponential generating function $\sum_n e(2n)x^{2n}/(2n)!$.

(b) Deduce that $e(2n) = (n!!)^2$.

5. (a) From the preceding problems it follows that the number of permutations of a $2n$ element set S with only even cycles is equal to the number of pairs of perfect matchings on S. Construct a direct bijection between the two (to do this you will probably need to fix the set S to be $[2n]$ and use the numerical values of its elements to make some auxiliary choices).

(b) Show that the species of permutations with even-length cycles and the species of pairs of perfect matchings are not equivalent. This can be understood as explaining the need for auxiliary choices in part (a).

6. (a) Find the exponential generating function $\sum_n r_n x^n/n!$, where r_n is the number of permutations of odd order of an n element set.

(b) Deduce that $r_{2n} = (n!!)^2$ and $r_{2n+1} = (2n+1)(n!!)^2$.

7. (a) The *diameter* d of a tree T is the maximum length of a path in T (a path of length n has n edges and $n+1$ vertices). Prove that if d is even then all paths of length d have the same middle vertex, called the *center* of T, and if d is odd, then all paths of length d have the same middle edge, called the *bicenter* of T.

(b) Show that if d is even, the species of labelled unrooted trees of diameter d is equivalent to the species of labelled rooted trees of height $d/2$ with the property that at least two children of the root are roots of subtrees of height $d/2 - 1$.

(c) Show that if d is odd, the species of labelled unrooted trees of diameter d is equivalent to the species of unordered pairs of disjoint rooted trees of height $(d-1)/2$.

(d) Let T_h be the species of labelled rooted trees of height h, and let $T_{\leq h} = T_0 + \cdots + T_h$. Show that these are given by the recurrence

$$\begin{aligned} T_0 &= X \\ T_h &= X((E-1)\circ T_{h-1})(E\circ T_{\leq h-2}) \quad \text{for } h > 0. \end{aligned}$$

(e) Use (a), (b) and (c) to express the species U_d of labelled unrooted trees of diameter d in terms the species T_h.

(f) Use (d) and (e) to calculate the number of labelled unrooted trees of diameter d on n vertices, for $n \leq 5$ and all d. Check that your answers summed over d agree with the known formula n^{n-2}.

(g) Use (d) and (e) to calculate the number of unlabelled unrooted trees of diameter d on n vertices, for $n \leq 5$ and all d. Check your answers by listing the trees.

8. The group of *signed permutations* B_n is a Coxeter group acting on $\mathbb{R}^n$. We can represent each $w \in B_n$ by the vector $(w_1, \ldots, w_n) = w \cdot (1, 2, \ldots, n)$. Thus $(w_1, \ldots, w_n)$ is a word in the letters $\{\pm 1, \ldots, \pm n\}$ such that the absolute values $|w_i|$ form a permutation.

Let us take as Coxeter generators the transpositions $\sigma_i = (i\; i{+}1) \in S_n$ and the sign change $\tau(x_1, x_2, \ldots, x_n) = (-x_1, x_2 \ldots, x_n)$. Define $\mathrm{inv}(w) = |\{i < j : |w_i| > w_j\}| + |\{i : w_i < 0\}|$.

(a) Show that $\mathrm{inv}(w)$ is the equal to the minimum length of an expression for w as a product of the Coxeter generators.

(b) Using Chevalley's theorem, prove that the ring of invariants $\mathbb{R}[x_1, \ldots, x_n]^{B_n}$ is generated by the even power-sums $p_{2k}(x_1, \ldots, x_n)$ for $k = 1, \ldots, n$. In particular, the invariant degrees are $2, 4, \ldots, 2n$.

(c) Derive the identity

$$\sum_{w \in B_n} q^{\mathrm{inv}(w)} = (2n)_q (2n-2)_q \cdots (2)_q.$$

in two ways: (i) as a consequence of Chevalley's theorem, and (ii) by a direct combinatorial argument.

(d) Describe all the reflections in B_n and verify in this example the general theorem that the number of reflections is the sum of the exponents (defined as $e_i = d_i - 1$, where d_i are the invariant degrees).