Math 249, Fall 2017
Problem Set 1

Due: Wednesday, Nov. 8

1. Recall that a sequence of formal power series $F_k(x) = \sum_n a_n^{(k)} x^n$ over a ring R converges to $G(x) = \sum_n b_n x^n$ if for each n the sequence $(a_n^{(1)}, a_n^{(2)}, \ldots)$ converges to b_n in the discrete topology, i.e., if $a_n^{(k)} = b_n$ for all sufficiently large k.

Assume that the coefficient ring R is commutative and has a unit element 1.

(a) Prove that the partial sums $a_0 + a_1 x + \cdots + a_n x^n$ converge to $\sum_n a_n x^n$.

(b) Prove that the sum $\sum_{k=1}^{\infty} F_k(x)$ converges if and only if the terms $F_k(x)$ converge to zero, and that this property and the value of the limit do not depend on the order of the terms.

(c) Prove that if the factors $F_k(x)$ converge to 1, then the product $\prod_{k=1}^{\infty} F_k(x)$ converges, and its value does not depend on the order of the factors. Prove also that if the constant term $a_0^{(k)}$ of $F_k(x)$ is a non-zero-divisor in R for every k, then $\prod_{k=1}^{\infty} F_k(x)$ converges if and only if the $F_k(x)$ converge to 1.

(d) Prove that if infinitely many of the factors $F_k(x)$ have zero constant term, or if any of the factors $F_k(x)$ is identically zero, then $\prod_{k=1}^{\infty} F_k(x)$ converges to zero, independent of the order of the factors.

(e) Assume that the coefficient ring R is integral domain. Prove that $\prod_{k=1}^{\infty} F_k(x)$ converges to zero only in the cases in part (d), and that it converges to a non-zero limit if and only if the $F_k(x)$ converge to 1 and none of them is identically zero.

(f) Prove that a sum or product of limits of convergent sequences is the limit of the sums or products term by term. Show that this also holds for convergent infinite sums and products.

2. Show that $F(x) \in R[[x]]$ has a multiplicative inverse if and only if its constant term $F(0)$ has a multiplicative inverse in R.

3. If $F(x)$ and $G(x)$ are formal power series and $F(0) = 0$, i.e., $F(x)$ has zero constant term, their formal composition is defined by

$$(G \circ F)(x) = \sum_{k=0}^{\infty} g_k F(x)^k,$$

where $G(x) = \sum_{k=0}^{\infty} g_k x^k$. Note that the sum converges by part (b) of the preceding problem. We also write $G(F(x))$ for $(G \circ F)(x)$.

(a) Show that composition is associative, i.e., if $F(0) = 0$ and $G(0) = 0$, then $(H \circ G) \circ F = H \circ (G \circ F)$.

(b) Show that composition with a fixed F, considered as a function of G, is a ring homomorphism.

(c) Show that $F(x) \in R[[x]]$ such that $F(0) = 0$ has a formal compositional inverse if and only if the coefficient $\langle x \rangle F(x)$ has a multiplicative inverse in R.

4. Assume we are working in a formal power series ring $R[[x]]$ over a coefficient ring R containing $\mathbb{Q}$. Define $\exp(x)$ and $\log(1+x)$ by their usual Taylor series.

(a) Given $F(x), G(x) \in R[[x]]$ such that $F(0) = 1$, show that there is a well-defined formal power series

$$F(x)^{G(x)} \underset{\text{def}}{=} \exp(G(x) \log F(x)).$$

(b) Show that the above definition satisfies the usual laws of exponents, namely, $F(x)^{G(x)+H(x)} = F(x)^{G(x)} F(x)^{H(x)}$, $F(x)^{G(x)H(x)} = (F(x)^{G(x)})^{H(x)}$, $F(x)^0 = 1$, $F(x)^1 = F(x)$.

5. Defining the formal derivative $\frac{d}{dx} F(x)$ of a formal power series term by term in the obvious way, show that the usual sum and product rules and the chain rule hold. Show that Taylor's formula holds if the coefficient ring contains $\mathbb{Q}$.

6. Express each of the following as a binomial coefficient: (a) the number of monomials of degree exactly d in a (commutative) polynomial ring in n variables; (b) the number of monomials of degree $\leq d$.

7. Find the number of monotone maps $f\colon \{1, \ldots, k\} \to \{1, \ldots, n\}$, where *monotone* means $f(i) \leq f(j)$ for $i \leq j$.

8. Show that $\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} \left\langle {n-1 \atop j} \right\rangle$. Using this, evaluate $\sum_{j=0}^{k} \binom{n+j}{j}$ as a single binomial coefficient.

9. (a) Give two proofs of the binomial coefficient identity, called the *convolution formula*,

$$\sum_{j} \binom{m}{j} \binom{n}{k-j} = \binom{m+n}{k}.$$

One proof should use generating functions, the other should be a direct combinatorial proof.

(b) Discover and prove in the same two ways an analogous identity for multiset coefficients $\left\langle {n \atop k} \right\rangle$.

10. Find a simple expression for the ordinary generating function in two variables

$$\sum_{n,k \geq 0} \binom{n}{k} x^n y^k,$$

and use it to deduce the identity

$$\sum_{r} \binom{r}{k} x^r = \frac{x^k}{(1-x)^{k+1}}.$$

11. Show that a finite group G is abelian if and only if all the irreducible representations of G (over $\mathbb{C}$) are one-dimensional.

12. Use characters to show that if V is an irreducible representation of G and W is an irreducible representation of H, then $V \otimes W$ is an irreducible representation of $G \times H$ (over $\mathbb{C}$, for finite groups).

13. Find the character tables of

(a) the dihedral group D_8 of order 8;

(b) the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, where $i^2 = j^2 = k^2 = ijk = -1$, and signs multiply according to the usual rules.

Show that Q_8 and D_8 are not isomorphic, and conclude that the character table of a finite group need not determine the group.

14. Let G be a finite group and $V_1, \ldots, V_k$ a list of its irreducible representations (one from each isomorphism class). Recall that the actions $\mathbb{C}G \to \mathrm{End}_{\mathbb{C}}(V_i)$ of $\mathbb{C}G$ on each V_i induce an isomorphism $\mathbb{C}G \cong \prod_i \mathrm{End}_{\mathbb{C}}(V_i)$. Let $e_i \in \mathbb{C}G$ be the element that acts as the identity on V_i and as zero on V_j for all $j \neq i$.

(a) Show that the elements e_i are a basis of the center of $\mathbb{C}G$.

(b) Use orthogonality of characters to derive the explicit formula

$$e_i = \frac{\dim(V_i)}{|G|} \sum_{g \in G} \chi_i(g^{-1})\, g,$$

where χ_i is the character of V_i. Hint: first verify that the formula gives an element in the center of $\mathbb{C}G$.

15. For $1 \leq i < j \leq n$, define the raising operator R_{ij} on $\mathbb{Z}^n$ by

$$R_{ij}(\nu_1, \ldots, \nu_n) = (\nu_1, \ldots, \nu_i + 1, \ldots, \nu_j - 1, \ldots, \nu_n).$$

(a) Show that the dominance order $\leq$ is the transitive closure of the relation on partitions $\lambda \to \mu$ if $\mu = R_{ij}\lambda$ for some $i < j$.

(b) We say that μ *covers* λ if $\lambda < \mu$ and there is no ν such that $\lambda < \nu < \mu$. Show that μ covers λ if and only if $\mu = R_{ij}\lambda$, where i, j satisfy the following condition: either $j = i + 1$, or $\lambda_i = \lambda_j$ (or both).

(c) Find the smallest n such that the dominance order on partitions of n is not a total ordering, and draw its Hasse diagram (i.e., the graph of the covering relation).

16. Show that the dominance partial order on partitions of n satisfies

$$\lambda \leq \mu \quad \Leftrightarrow \quad \lambda^* \geq \mu^*,$$

where λ^* denotes the transpose partition of λ.

17. Prove the formula for complete homogeneous symmetric functions in terms of elementary symmetric functions

$$h_n = \sum_{|\lambda| = n} (-1)^{n - l(\lambda)} \binom{l(\lambda)}{r_1, r_2, \ldots, r_k} e_\lambda,$$

where $\lambda = (1^{r_1}, 2^{r_2}, \dots, k^{r_k})$.

18. We know that if $\mu \not\leq \lambda$, then $K_{\lambda\mu} = 0$, where $K_{\lambda\mu} = \langle s_\lambda, h_\mu \rangle = |SSYT(\lambda, \mu)|$ is the number of semi-standard tableaux of shape λ and weight μ. Prove that, conversely, if $\mu \leq \lambda$, then $K_{\lambda\mu} \neq 0$.

19. The symmetric functions $f_\lambda = \omega m_\lambda$ are sometimes called the "forgotten" symmetric functions. Show that the matrix of coefficients of the forgotten symmetric functions f_λ expressed in terms of monomial symmetric functions m_λ is the transpose of the matrix of the elementary symmetric functions e_λ expressed in terms of the complete homogeneous symmetric functions h_λ.

20. For any symmetric polynomial f, let $f^\perp$ be the operator adjoint to multiplication by f with respect to the Hall inner product, that is, $\langle f^\perp g, h \rangle = \langle g, fh \rangle$ for all g, h.

(a) Find a formula for $h_k^\perp m_\lambda$, expressed again in terms of monomial symmetric functions m_μ.

(b) Show that the basis of monomial symmetric functions is uniquely characterized by the formula from part (a).

21. Let ∂p_k be the operator on symmetric functions given by partial differentiation with respect to p_k, under the identification of the algebra of symmetric functions with the polynomial ring $\mathbb{Q}[p_1, p_2, \dots]$. Show that ∂p_k is adjoint with respect to the Hall inner product to the operator of multiplication by p_k/k.

22. [From I. G. Macdonald, Symmetric Functions and Hall Polynomials] (a) Recall from class that $h_n = \sum_{|\lambda|=n} p_\lambda / z_\lambda$, where $z_\lambda = \prod_i i^{r_i} r_i!$ for $\lambda = (1^{r_1}, 2^{r_2}, \dots)$. Show that this is equivalent to Newton's determinant formula

$$h_n = \frac{1}{n!} \det \begin{bmatrix} p_1 & -1 & 0 & \dots & 0 \\ p_2 & p_1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n-1} & p_{n-2} & . & \dots & -(n-1) \\ p_n & p_{n-1} & . & \dots & p_1 \end{bmatrix}$$

(b) Show that e_n is given by the same determinant without the minus signs.

23. [From Macdonald] Prove the identity $s_{(n-1,n-2,\dots,1)}(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_i + x_j)$.

24. [From Macdonald] $|\lambda| = |\mu| = n$. Show that $\langle h_\lambda, h_\mu \rangle$ is equal to the number of double cosets $S_\lambda w S_\mu$ in the symmetric group S_n, where S_λ and S_μ are Young subgroups of S_n.

25. Prove that the Frobenius characteristic map is given in terms of monomial symmetric functions by

$$F(\chi_V) = \sum_\mu \dim(V^{S_\mu})\, m_\mu,$$

where χ_V is the character of an S_n module V, S_μ is the Young subgroup $S_{\mu_1} \times \cdots \times S_{\mu_l} \subseteq S_n$, and $V^{S_\mu} \subseteq V$ denotes the subspace of elements invariant under the action of S_μ.

26. Let $\lambda = (l^k)$ and $\mu = (n^m)$ be partitions whose diagrams are rectangular. Suppose that $k \leq m$ and $l \leq n$, that is, the diagram of λ is contained in that of μ. Prove that $s_\lambda s_\mu = \sum_\nu s_\nu$, where ν ranges over partitions whose diagram contains the $m \times n$ rectangular diagram of μ, and the portion of the diagram of ν outside this rectangle consists of a diagram α on top of the rectangle and a diagram β to the right of the rectangle, such that α and the $180°$ rotation of β fit together in a $k \times l$ rectangle.