Math 249 Fall 2016—Problem Set 3

You may submit problems from this set on paper at my office through Tuesday, Dec. 13. The following Wednesday morning will be my last visit to the office before winter break.

You may submit problems by email up to Friday, Dec. 16.

I will probably read Problem Sets 1 and 2 during Reading Week. I will accept late submissions received before then.

1. Let GL_n act by conjugation on the space M_n of $n \times n$ matrices. Show that the character of this representation of GL_n is $(x_1 + \cdots + x_n)(x_1^{-1} + \cdots + x_n^{-1})$. Expand this in terms of irreducible characters $(x_1 \cdots x_n)^{-k} s_\lambda(x)$, where s_λ denotes a Schur function. Describe a decompostion of M_n as a direct sum of invariant subspaces corresponding to the terms in the resulting expansion.

2. Prove the formula for complete homogeneous symmetric functions in terms of elementary symmetric functions

$$h_n = \sum_{|\lambda|=n} (-1)^{n-l(\lambda)} \binom{l(\lambda)}{r_1, r_2, \ldots, r_k} e_\lambda,$$

where $\lambda = (1^{r_1}, 2^{r_2}, \ldots, k^{r_k})$.

3. Let ϵ be a fictitious alphabet such that $p_k[\epsilon] = \delta_{1,k}$. Stated more correctly, this means we are to interpret $f[\epsilon]$ as the image of f under the homomorphism $\Lambda \to \mathbb{Q}$ mapping p_k to $\delta_{1,k}$.

(a) Prove the identity $f[\epsilon] = \langle f, \exp(p_1) \rangle$.

(b) Prove the identity $f[\epsilon] = \lim_{n\to\infty} (f[nx])_{x \mapsto 1/n}$.

(c) Show that $e_k[\epsilon] = h_k[\epsilon] = 1/n!$.

(d) More generally, show that $s_\lambda[\epsilon] = f_\lambda/n!$, where $|\lambda| = n$ and f_λ is the number of standard Young tableaux of shape λ.

4. (a) Recall from class that $h_n = \sum_{|\lambda|=n} p_\lambda / z_\lambda$, where $z_\lambda = \prod_i i^{r_i} r_i!$ for $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$. Show that this is equivalent to Newton's determinant formula

$$h_n = \frac{1}{n!} \det \begin{bmatrix} p_1 & -1 & 0 & \ldots & 0 \\ p_2 & p_1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n-1} & p_{n-2} & . & \ldots & -(n-1) \\ p_n & p_{n-1} & . & \ldots & p_1 \end{bmatrix}$$

(b) Show that e_n is given by the same determinant without the minus signs.

[From I. G. Macdonald, Symmetric Functions and Hall Polynomials]

5. Let $\lambda = (k, 1^l)$ be a hook shape. Prove the following rule for computing $s_\lambda s_\mu$, which generalizes the Pieri rules for $s_{(k)} s_\mu$ and $s_{(1^k)} s_\mu$:

(i) s_ν occurs with non-zero coefficient in $s_\lambda s_\mu$ only if ν/μ is a disjoint union of ribbons (connected skew shapes containing no 2×2 rectangle) of total size $k + l$,

(ii) in that case, the coefficient is $\binom{r-1}{h-l-1}$, where μ/ν consists of r ribbons, and h is the sum of their heights (interpreting $\binom{r-1}{h-l-1}$ as zero if $h - l - 1$ is negative).

6. Consider the partition (a^b) whose diagram is an $a \times b$ rectangle. Prove that

$$s_{(a^b)}^2 = \sum_\mu s_\mu,$$

where all coefficients are equal to 1 and the sum ranges over partitions μ containing (a^b) and such that $\mu/(a^b)$ is the disjoint union of (translates of) partition diagrams $\rho, \nu \subseteq (a^b)$, satisfying $\nu = ((a^b)/\rho)^\perp$. Here $(-)^\perp$ denotes rotation of a diagram through 180°.

7. Compute the character table of S_5.

8. (a) Prove that $V_{(n-1,1)}$ is the irreducible submodule of dimension of dimension $n-1$ in the defining representation of S_n on $\mathbb{C}^n$.

(b) Prove that $V_{(n-k,1^k)}$ is isomorphic to the k-th exterior power of $V_{(n-1,1)}$.

9. The *Kronecker product* on symmetric functions is defined in terms of the power-sum basis by

$$p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda.$$

Equivalently, the symmetric functions p_λ / z_λ are orthogonal idempotents with respect to $*$.

(a) Show that $s_\mu * s_\nu$ is the Frobenius characteristic of the character of the tensor product $V_\mu \otimes V_\nu$ of the irreducible S_n modules with characters χ_μ and χ_ν, where $|\mu| = |\nu| = n$ (if $|\mu| \neq |\nu|$, then $s_\mu * s_\nu = 0$).

(b) Deduce that the Kronecker coefficients $a_{\lambda,\mu,\nu}$ defined by

$$s_\mu * s_\nu = \sum_\lambda a_{\lambda,\mu,\nu} s_\lambda$$

are non-negative integers. It is an open problem to find a general combinatorial rule for these coefficients.

(c) Show that the Kronecker coefficients are also given by

$$s_\lambda[XY] = \sum_{\mu,\nu} a_{\lambda,\mu,\nu} s_\mu[X] s_\nu[Y].$$

More generally, show that the coefficients in the expansion of $s_\lambda[X_1 \dots X_k]$ in terms of products $s_{\mu_1}[X_1] \cdots s_{\mu_1}[X_1]$ are the same as the coefficients of Schur functions s_λ in the Kronecker product $s_{\mu_1} * \cdots * s_{\mu_k}$.

(d) Show that

$$\Omega[XYZ] = \sum_{\lambda,\mu,\nu} a_{\lambda,\mu,\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z].$$

In particular, $a_{\lambda,\mu,\nu}$ is symmetric in all three indices.