Math 249 Fall 2016—Problem Set 2

1. Prove that the number of partitions of n with no parts divisible by d is equal to the number of partitions of n with no part repeated d or more times, for all n and d.

2. Let $p_+(n)$ be the the number of partitions of n with an even number of parts and $p_-(n)$ the number with an odd number of parts. Let $p_{DO}(n)$ be the number of partitions of n with distinct odd parts, and let $k(n)$ be the number of partitions λ of n such that $\lambda = \lambda^*$. Prove that

$$k(n) = p_{DO}(n) = (-1)^n (p_+(n) - p_-(n)).$$

Hint: if $\lambda = \lambda^*$, dissect the diagram of λ into 'hooks' whose sizes are odd and distinct.

3. The following two identities are due to Gauss:

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} = \prod_{i\geq 1} \frac{1-q^i}{1+q^i};$$

$$\sum_{n\geq 0} q^{\binom{n+1}{2}} = \prod_{i\geq 1} \frac{1-q^{2i}}{1-q^{2i-1}}.$$

(a) Interpret them combinatorially as partition identities.

(b) Prove them, either combinatorially (not so easy) or using Jacobi's triple product identity.

4. The *Durfee square* of a partition λ is the largest $k \times k$ square that fits inside its Young diagram. Use Durfee squares to prove the following identities [Stanley, 2nd Ed., Proposition 1.8.6(b) and Ex. 1.76]:

(a)

$$\prod_{i\geq 1} \frac{1}{1-tx^i} = \sum_{k\geq 0} \frac{t^k x^{k^2}}{(1-x)\cdots(1-x^k)\,(1-tx)\cdots(1-tx^k)}$$

(b)

$$\prod_{i\geq 1} 1 + tx^{2i-1} = \sum_{k\geq 0} \frac{t^k x^{k^2}}{(1-x^2)\cdots(1-x^{2k})}$$

5. (a) Use Durfee squares to obtain an identity analogous to part (a) of the previous problem, but enumerating partitions with distinct parts.

(b) Deduce *Sylvester's identity*

$$\prod_{i\geq 1}(1-tq^i) = 1 + \sum_{n\geq 1} (-1)^n t^n \left(q^{(3n^2+n)/2} \prod_{i=1}^{n} \frac{1-tq^i}{1-q^i} + q^{(3n^2-n)/2} \prod_{i=1}^{n-1} \frac{1-tq^i}{1-q^i} \right),$$

a generalization of Euler's pentagonal number theorem.

6. Show that the Stirling numbers of the second kind $S(n,k)$ have a q-analog $S_q(n,k)$ characterized by any of the following properties, where $[k]_q = (1-q^k)/(1-q) = 1+q+\cdots+q^{k-1}$ is the q analog of k.

(a) They satisfy the recurrence

$$S_q(n,k) = [k]_q S_q(n-1,k) + q^{k-1} S_q(n-1,k-1),$$

with inital condition $S_q(0,n) = S_q(n,0) = \delta_{0,n}$.

(b) They satisfy the following q-analog of the classical formula $x^n = \sum_k S(n,k)(x)_k$:

$$[r]_q^n = \sum_k S_q(n,k)[r]_q[r-1]_q \cdots [r-k+1]_q$$

(c) For each k, they are given by the ordinary generating function

$$\sum_n S_q(n,k)x^n = \frac{q^{\binom{k}{2}} x^k}{(1-x)(1-[2]_q x)\cdots(1-[k]_q x)}.$$

(d) Given a partition $\pi = \{B_1, \ldots, B_k\}$ of $[n]$, with the blocks numbered so that $\min(B_i) < \min(B_j)$ for $i < j$, define $\nu(\pi) = \sum_i (i-1)|B_i|$. Then $S_q(n,k) = \sum_\pi q^{\nu(\pi)}$, where the sum is over partitions of $[n]$ into k blocks.

7. One way to define a q-analog of the Eulerian polynomial $A_n(x)$ is

$$A_n(x,q) = \sum_{\sigma \in S_n} x^{d(\sigma)+1} q^{\mathrm{maj}(\sigma)}.$$

(a) Show that with this definition we have

$$\sum_r [r]_q^n x^r = \frac{A_n(x,q)}{(1-x)(1-qx)\cdots(1-q^n x)}$$

(b) Deduce the formula

$$A_n(x,q) = \sum_k [k]_q! S_q(n,k) x^k \prod_{i=k+1}^{n} (1-xq^i),$$

where $S_q(n,k)$ are the q-analogs of Stirling numbers defined in the previous problem.

8. Define the descent set of a word $w \in \mathbb{N}^n$ to be $D(w) = \{i \in [n-1] : w(i) > w(i+1)\}$, just as one does for permutations. Similarly, define $\mathrm{maj}(w) = \sum_{d \in D(w)} d$.

(a) Show that if $w \in [r]^n$ and $w(n) = s$, then $w' = (1\ 2\ \cdots\ r)^{r-s} \circ w$ has $\mathrm{maj}(w') = \mathrm{maj}(w) - (k_{s+1} + \cdots + k_r)$, where k_i is the number of occurences of i in the word w.

(b) Use (a) and the recurrence for q-multinomial coefficients in Problem Set 1, Problem 11 to prove that

$$\sum_{w \in S_n \cdot (1^{k_1}, 2^{k_2}, \dots, r^{k_r})} t^{\mathrm{maj}(w)} = \sum_{w \in S_n \cdot (1^{k_1}, 2^{k_2}, \dots, r^{k_r})} t^{\mathrm{inv}(w)},$$

for all $k_1 + \cdots + k_r = n$.

(c) Use (b) to prove that for all $D \subseteq [n-1]$,

$$\sum_{\substack{\pi \in S_n \\ D(\pi^{-1}) = D}} t^{\mathrm{maj}(\pi)} = \sum_{\substack{\pi \in S_n \\ D(\pi^{-1}) = D}} t^{\mathrm{inv}(\pi)},$$

that is, inv and maj are equidistributed on inverse descent classes.

(d) Deduce that

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{maj}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi^{-1})} t^{\mathrm{maj}(\pi)}.$$

This gives a different proof than the one we did in class that the sum on the left-hand side is symmetric in q and t.

Remark: part (c) implies that $\sum_{\pi \in S_n} x^{d(\pi)+1} q^{\mathrm{maj}(\pi^{-1})} = \sum_{\pi \in S_n} x^{d(\pi)+1} q^{\mathrm{inv}(\pi)}$, which suggests that the common value of these two expressions might be a 'better' q-analog of $A_n(x)$ than the one Problem 7. However, I don't know of nice identities for this alternative q-Eulerian polynomial like those in Problem 7.

9. A *distribution* is a function together with a linear ordering on the preimage of each element of the codomain. Use exponential generating functions to obtain the formula

$$(n)_k (n-1)_{n-k}.$$

for the number of surjective distributions from a set of n labelled objects to a set of k labelled places.

10. An *at most binary tree* is an unordered rooted tree in which each node has at most two children.

(a) Find the exponential generating function for the number of at most binary trees on n labelled nodes.

(b) Find the ordinary generating function for the number of at most binary trees on n unlabelled nodes.

In the next three problems we use the 'double factorial' notation

$$n!! = (2n-1)(2n-3)\cdots 3 \cdot 1.$$

11. Use the exponential generating function for the species of perfect matchings to show that the number of perfect matchings on a set of size $2n$ is $n!!$ (as seen by a direct counting argument in Problem Set 1, Problem 15(a))

12. Let e_{2n} be the number of permutations $w \in S_{2n}$ such that every cycle of w has even length.

(a) Find the exponential generating function $\sum_n e_{2n} x^{2n}/(2n)!$, and deduce that $e_{2n} = (n!!)^2$.

(b) It follows that if S is a set of size $2n$, the number of permutations of S with only even cycles is equal to the number of pairs of perfect matchings on S. Construct a direct bijection between the two. You will probably need to fix S to be $\{1, 2, \ldots, 2n\}$ and use the numerical values of its elements to make some auxiliary choices.

(c) Show that the species of permutations with even-length cycles and the species of pairs of perfect matchings are not equivalent. This can be understood as explaining the need for auxiliary choices in part (b).

13. Find the exponential generating function $\sum_n r_n x^n/n!$, where r_n is the number of permutations of odd order of an n element set. Deduce that $r_{2n} = (n!!)^2$ and $r_{2n+1} = (2n+1)(n!!)^2$.

14. Let $g_+(n)$, $g_-(n)$ denote the number of connected simple graphs on the vertex set $[n]$ with an even or an odd number of edges, respectively. Prove that $g_+(n) - g_-(n) = (-1)^n (n-1)!$.

15. As part of the computer demonstration in class, we counted unrooted trees by counting connected graphs with n vertices and e edges, then picking out the terms with $e = n - 1$. In this problem, we outline a different method of counting unrooted trees.

(a) The *diameter* d of a tree T is the maximum length of a path in T (a path of length n has n edges and $n+1$ vertices). Prove that if d is even then all paths of length d have the same middle vertex, called the *center* of T, and if d is odd, then all paths of length d have the same middle edge, called the *bicenter* of T.

(b) Show that if d is even, the species of labelled unrooted trees of diameter d is equivalent to the species of labelled rooted trees of height $d/2$ with the property that at least two children of the root are roots of subtrees of height $d/2 - 1$.

(c) Show that if d is odd, the species of labelled unrooted trees of diameter d is equivalent to the species of unordered pairs of disjoint rooted trees of height $(d-1)/2$.

(d) Let T_h be the species of labelled rooted trees of height h, and let $T_{\leq h} = T_0 + \cdots + T_h$. Show that these are given by the recurrence

$$\begin{aligned} T_0 &= X \\ T_h &= X((E-1) \circ T_{h-1})(E \circ T_{\leq h-2}) \quad \text{for } h > 0. \end{aligned}$$

(e) Use (a), (b) and (c) to express the species U_d of labelled unrooted trees of diameter d in terms the species T_h.

(f) Use (d) and (e) to calculate the number of labelled unrooted trees of diameter d on n vertices, for $n \leq 5$ and all d. Check that your answers summed over d agree with the known formula n^{n-2}.

(g) Use (d) and (e) to calculate the number of unlabelled unrooted trees of diameter d on n vertices, for $n \leq 5$ and all d. Check your answers by listing the trees.

16. (a) Find an explicit formula for the cycle index Z_I of the species of involutions, $I(S) = \{\sigma \in \mathfrak{S}_S : \sigma^2 = 1\}$.

(b) Evaluate $Z_I[x]$ and verify that it agrees with the obvious ordinary generating function counting involutions up to conjugacy.

17. (a) From Cayley's tree generating function derive the identity

$$\sum_F \prod_{i=1}^{n} x_i^{c_F(i)} = \binom{n-1}{k-1}(x_1 + \cdots x_n)^{n-k},$$

where the sum is over rooted forests F with k components on vertices $\{1, \ldots, n\}$, and $c_F(i)$ denotes the number of children of vertex i in F.

(b) Deduce the identity

$$\sum_F \prod_{i=1}^{n} h_{c_F(i)} = \binom{n-1}{k-1}\langle \frac{x^{n-k}}{(n-k)!}\rangle H(x)^n,$$

where $H(x) = \sum_m h_m x^m/m!$ is a generic formal power series written in exponential form and $\langle \cdot \rangle$ denotes taking a coefficient.

(c) Let H be the 'generic species' with exponential generating function $H(x)$, that is, the trivial species, but enumerated by assigning weight h_n to the one structure on any set with n elements. Let $F(x)$ be the solution of the formal functional equation

$$F(x) = x\, H(F(x))$$

(that is, assuming h_0 invertible, the functional composition inverse of $x/H(x)$). Using (b) and the species interpretation of $F(x)$, obtain the generalized Lagrange inversion formula

$$\langle \frac{x^n}{n!}\rangle F(x)^k/k! = \binom{n-1}{k-1}\langle \frac{x^{n-k}}{(n-k)!}\rangle H(x)^n,$$

or equivalently,

$$\langle x^n \rangle F(x)^k = \frac{k}{n}\langle x^{n-k}\rangle H(x)^n$$

(we did the case $k = 1$ in class).

18. Prove that the number of ordered rooted trees with $n+1$ vertices and j leaves is equal to

$$\frac{1}{n+1}\binom{n+1}{j}\binom{n-1}{n-j}.$$

19. The product $G \times H$ of two simple graphs (graphs without loops or multiple edges) is defined to be the graph on vertex set $V(G) \times V(H)$ with edges $\{(v, w), (v', w')\}$ for $v = v'$

and $\{w, w'\} \in E(H)$ or $w = w'$ and $\{v, v'\} \in E(G)$. The adjacency matrix A_G of a graph G on n vertices is the $n \times n$ matrix with rows and columns labelled by the vertices, and entries $(A_G)_{v,w} = 1$ if $\{v, w\} \in E(G)$, zero otherwise. Let D_G be the diagonal matrix whose (v, v) entry is the degree of v.

(a) Let $f_G(r)$ be the number of rooted spanning forests of G with r roots, and let $F_G(z) = \sum_r f_G(r)z^r$ be the corresponding generating function. Show that $F_G(z) = \prod_i (z+\alpha_i)$, where the α_i's are the eigenvalues of $D_G - A_G$.

(b) Show that $F_{G\times H}(z) = \prod_{i,j}(z + \alpha_i + \beta_j)$, where $F_G(z) = \prod_i (z + \alpha_i)$ and $F_H(z) = \prod_j (z+\beta_j)$. In particular, the numbers $f_G(r)$ and $f_H(r)$ for all r determine the corresponding numbers $f_{G\times H}(r)$.

(c) Show that if Q_n is the graph formed by the vertices and edges of the n-cube, that is, the product of n copies of the complete graph on 2 vertices, then

$$F_{Q_n}(z) = \prod_{k=0}^{n} (z + 2k)^{\binom{n}{k}}.$$

This generalizes Stanley, Exercise 5.6.10, which follows by taking the coefficient of z.

20. Let $f_m(r)$ be the number of rooted spanning forests with r roots in the graph C_m, a cycle on m vertices ($m > 1$). Prove that

$$F_m(z) \underset{\text{def}}{=} \sum_r f_m(r)z^r = \prod_{j=0}^{m-1} (z + 2 - 2\cos(2\pi j/m)) = \sum_{r=1}^{m} \frac{m}{r}\binom{m+r-1}{2r-1} z^r.$$