Math 249 Fall 2016—Problem Set 1

1. Recall that a sequence of formal power series $F_k(x) = \sum_n a_n^{(k)} x^n$ over a ring R converges to $G(x) = \sum_n b_n x^n$ if for each n the sequence $(a_n^{(1)}, a_n^{(2)}, \ldots)$ converges to b_n in the discrete topology, i.e., if $a_n^{(k)} = b_n$ for all sufficiently large k.

Assume that the coefficient ring R is commutative and has a unit element 1.

(a) Prove that the partial sums $a_0 + a_1 x + \cdots + a_n x^n$ converge to $\sum_n a_n x^n$.

(b) Prove that the sum $\sum_{k=1}^{\infty} F_k(x)$ converges if and only if the terms $F_k(x)$ converge to zero, and that this property and the value of the limit do not depend on the order of the terms.

(c) Prove that if the factors $F_k(x)$ converge to 1, then the product $\prod_{k=1}^{\infty} F_k(x)$ converges, and its value does not depend on the order of the factors. Prove also that if the constant term $a_0^{(k)}$ of $F_k(x)$ is a non-zero-divisor in R for every k, then $\prod_{k=1}^{\infty} F_k(x)$ converges if and only if the $F_k(x)$ converge to 1.

(d) Prove that if infinitely many of the factors $F_k(x)$ have zero constant term, or if any of the factors $F_k(x)$ is identically zero, then $\prod_{k=1}^{\infty} F_k(x)$ converges to zero, independent of the order of the factors.

(e) Assume that the coefficient ring R is integral domain. Prove that $\prod_{k=1}^{\infty} F_k(x)$ converges to zero only in the cases in part (d), and that it converges to a non-zero limit if and only if the $F_k(x)$ converge to 1 and none of them is identically zero.

(f) Prove that a sum or product of limits of convergent sequences is the limit of the sums or products term by term. Show that this also holds for convergent infinite sums and products.

2. If $F(x)$ and $G(x)$ are formal power series and $F(0) = 0$, i.e., $F(x)$ has zero constant term, their formal composition is defined by

$$(G \circ F)(x) = \sum_{k=0}^{\infty} g_k F(x)^k,$$

where $G(x) = \sum_{k=0}^{\infty} g_k x^k$. Note that the sum converges by part (b) of the preceding problem. We also write $G(F(x))$ for $(G \circ F)(x)$.

(a) Show that composition is associative, i.e., if $F(0) = 0$ and $G(0) = 0$, then $(H \circ G) \circ F = H \circ (G \circ F)$.

(b) Show that composition with a fixed F, considered as a function of G, is a ring homomorphism.

(c) Show that $F(x) \in R[[x]]$ such that $F(0) = 0$ has a formal compositional inverse if and only if the coefficient $\langle x \rangle F(x)$ has a multiplicative inverse in R.

3. Show that $F(x) \in R[[x]]$ has a multiplicative inverse if and only if its constant term $F(0)$ has a multiplicative inverse in R.

4. Assume we are working in a formal power series ring $R[[x]]$ over a coefficient ring R containing $\mathbb{Q}$. Define $\exp(x)$ and $\log(1+x)$ by their usual Taylor series.

(a) Given $F(x), G(x) \in R[[x]]$ such that $F(0) = 1$, show that there is a well-defined formal power series

$$F(x)^{G(x)} \underset{\text{def}}{=} \exp(G(x) \log F(x)).$$

(b) Show that the above definition satisfies the usual laws of exponents, namely, $F(x)^{G(x)+H(x)} = F(x)^{G(x)} F(x)^{H(x)}$, $F(x)^{G(x)H(x)} = (F(x)^{G(x)})^{H(x)}$, $F(x)^0 = 1$, $F(x)^1 = F(x)$.

5. Express each of the following as a binomial coefficient: (a) the number of monomials of degree exactly d in a (commutative) polynomial ring in n variables; (b) the number of monomials of degree $\leq d$.

6. Find the number of monotone maps $f\colon \{1, \ldots, k\} \to \{1, \ldots, n\}$, where *monotone* means $f(i) \leq f(j)$ for $i \leq j$.

7. (a) Give two proofs of the binomial coefficient identity, called the *convolution formula*,

$$\sum_j \binom{m}{j} \binom{n}{k-j} = \binom{m+n}{k}.$$

One proof should use generating functions, the other should be a direct combinatorial proof.

(b) Discover and prove in the same two ways an analogous identity for multiset coefficients $\left\langle {n \atop k} \right\rangle$.

8. Find a simple expression for the ordinary generating function in two variables

$$\sum_{n,k \geq 0} \binom{n}{k} x^n y^k,$$

and use it to deduce the identity

$$\sum_r \binom{r}{k} x^r = \frac{x^k}{(1-x)^{k+1}}.$$

9. Show that $\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} \left\langle {n-1 \atop j} \right\rangle$. Using this, evaluate $\sum_{j=0}^{k} \binom{n+j}{j}$ as a single binomial coefficient.

10. Regarding $\binom{x}{k}$ as a polynomial of degree k in the variable x, prove that a polynomial $f \in \mathbb{Q}[x]$ has the property that $f(n)$ is an integer for all integers n if and only if the coefficients of f with respect to the basis $\{\binom{x}{k} : k \in \mathbb{N}\}$ are integers.

Hint: for "only if," express the coefficients a_k such that $f(x) = \sum_k a_k \binom{x}{k}$ in terms of the iterated differences $(\Delta^m f)(0)$, where $\Delta f(x) = f(x+1) - f(x)$.

11. Prove that the q-multinomial coefficients satisfy the following recurrence. A combinatorial proof is preferred.

$$\binom{n}{k_1, k_2, \ldots, k_r}_q = \binom{n-1}{k_1 - 1, k_2, \ldots, k_r}_q + q^{k_1} \binom{n-1}{k_1, k_2 - 1, \ldots, k_r}_q + \cdots + q^{k_1 + \cdots + k_{r-1}} \binom{n-1}{k_1, k_2, \ldots, k_r - 1}_q.$$

12. Prove the following q-analog of the convolution formula for binomial coefficients. A combinatorial proof is preferred.

$$\binom{m+n}{k}_q = \sum_{i+j=k} q^{(m-i)j} \binom{m}{i}_q \binom{n}{j}_q$$

13. (a) Show that for each k there is a unique polynomial $Q_k(x)$ of degree k, with coefficients in the field of rational functions $\mathbb{Q}(q)$, such that $Q_k(q^n) = \binom{n}{k}_q$ for all n.

(b) Prove that a polynomial $f \in \mathbb{Q}(q)[x]$ has the property that $f(q^n) \in \mathbb{Z}[q, q^{-1}]$ for all n if and only if the coefficients of f with respect to the basis $\{Q_k : k \in \mathbb{N}\}$ belong to $\mathbb{Z}[q, q^{-1}]$. Hint: evaluate f at roots of the polynomials $Q_k(x)$.

14. Construct a formula for the generating function $\sum_n a_n x^n$, where a_n is the number of combinations of states and the District of Columbia having a combined number n of electoral votes (you can find the electoral vote distribution on the Wikipedia "Electoral College" page). Using a computer algebra system, evaluate the generating function explicitly and answer the following question: in a presidential election with two candidates, how many combinations lead to a tie in the electoral college, and how many to a win for either candidate?

To get it precisely right, you should take into account that Maine and Nebraska have district systems allowing them to split their electoral votes (Maine's 4 votes cannot split 2 and 2, however).

A news reporter contacted the MIT math department with this question when I was a grad student there. The department chair passed the problem along to us combinatorics grad students to solve.

15. A *perfect matching* on a set S of $2n$ elements is a partition of S into n blocks of 2 elements each. Taking $S = [2n] = \{1, 2, \cdots, 2n\}$, and thinking of the blocks in a matching as the edges of a graph, call edges of the form $\{i, i+1\}$ *short*, and all other edges *long*.

(a) Show that the number of perfect matchings on a $2n$-element set is

$$(2n-1)(2n-3)\cdots 3 \cdot 1.$$

(b) Let $M_n(x)$ be the ordinary generating function that counts perfect matchings on $[2n]$ with weight x^s, where s is the number of short edges, so for instance $M_2(x) = 1 + x + x^2$. Prove the recurrence

$$M_n(x) = (x + 2n - 2)M_{n-1}(x) + (1 - x)\frac{d}{dx}M_{n-1}(x).$$

16. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

(a) Derive the ordinary generating function

$$\sum_n F_n x^n = \frac{x}{1 - x - x^2}.$$

(b) By finding the roots of $1 - x - x^2 = 0$ and expanding (a) in partial fractions, obtain the explicit formula

$$F_n = \frac{1}{\sqrt{5}} \left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n \right)$$

(c) Prove that the number of subsets $S \subseteq \{1, \ldots, n\}$ such that no two elements of S are consecutive is F_{n+2}.

(d) Prove the identity

$$F_{n+2} = \sum_k \binom{n-k+1}{k}$$