1. \( \frac{\cos^2(n)}{n^2+1} \) is not decreasing, so the Integral Test does not apply.

2. \( \sum_{n=0}^{\infty} \frac{\cos^2(n)}{n^2+1} \) has positive terms, and \( \frac{\cos^2(n)}{n^2+1} \leq \frac{1}{n^2+1} < \frac{1}{n^2} \), so the series is convergent by comparison with the convergent p-series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

3. \( \sum_{n=3}^{\infty} \frac{1}{(n-1)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \) is the same as \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) minus its first term \( \frac{1}{1^2} = 1 \). Therefore \( \sum_{n=3}^{\infty} \frac{1}{(n-1)^2} = \frac{\pi^2}{6} - 1 \).

4. By the remainder estimate in the integral test, the remainder \( \sum_{k=n+1}^{\infty} \frac{1}{k^2} \) after the \( n \)-th partial sum is less than \( \int_{n}^{\infty} \frac{1}{x^2} \, dx = -\frac{1}{x} \bigg|_{n}^{\infty} = \frac{1}{n} \leq \frac{1}{3n^3} \).

We want \( n \) large enough to make \( \frac{1}{3n^3} < 0.01 \), i.e. \( 3n^3 > 100 \). The smallest such \( n \) is \( n = 4 \).

5. a) Abs. Conv. by limit comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) (or direct comparison).

   b) \( \left| \frac{n \sin(n)}{n^3+1} \right| \leq \frac{n}{n^3+1} < \frac{1}{n^2} \), hence Abs. Conv. by direct comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

   c) Divergent by limit comparison with \( \sum_{n=1}^{\infty} \frac{1}{n} \).

   d) Divergent because the terms \( (-1)^n \arctan(n) \) do not \( \to 0 \) as \( n \to \infty \) (note: \( \arctan(n) \to \frac{\pi}{2} \) as \( n \to \infty \)).

   c) Abs. Conv. by ratio test.
f) Convergent by alternating series test. The absolute values give a divergent series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n+5}$, by limit comparison with $\frac{1}{2} \frac{1}{\sqrt{n}}$, so the original series is only conditionally convergent, not abs. conver.

g) Equal to $\sum_{n=1}^{\infty} \frac{1}{2^n}$, convergent as a geometric series or log ratio test.

h) Divergent by ratio test.

6. If $p > 0$, it's divergent by comparison with the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$. If $p < 0$, the terms are decreasing and we can use the integral test.

$$\int_{\frac{1}{2}}^{\infty} \frac{(\ln x)^p}{x} \, dx = \int_{\ln 2}^{\infty} u^p \, du = \left[ \frac{u^{p+1}}{p+1} \right]_{\ln 2}^{\infty}$$

$$u = \ln x \quad (\text{or } \ln(u))$$

The improper integral is divergent if $p \leq 0$, convergent if $p > 0$, so the series is also divergent if $p \leq 0$, convergent if $p > 0$.

7. By the ratio test the series is convergent for $|2(x+1)| < 1$, divergent for $|2(x+1)| > 1$, i.e. convergent on $(-\frac{3}{2}, -\frac{1}{2})$, divergent outside of $[-\frac{3}{2}, -\frac{1}{2}]$.

At $x = -\frac{3}{2}$, we get $\sum_{n=0}^{\infty} \frac{1}{n+1}$, which is divergent (Harmonic series).

At $x = -\frac{1}{2}$, we get $\sum_{n=0}^{\infty} (-1)^n n+1$, which is a convergent alternating series. So the interval of convergence is $(-\frac{3}{2}, -\frac{1}{2}]$, radius of convergence $\frac{1}{2}$. 
8. Ratio test gives convergence for $|3x-4| < 1$, divergence for $|3x-4| > 1$, i.e. convergent on $(1, 5/3)$, divergent outside of $[1, 5/3]$. The radius of convergence is $3/2$. At $x = 5/3$, we get $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is convergent. At $x = 1$, we get $\sum_{n=1}^{\infty} \frac{2^{-n}}{n^2}$, which is also convergent, since $\left| \frac{(-1)^n}{n^2} \right| = \frac{1}{n^2}$.

The interval of convergence is $[1, 5/3]$.

9. (a) and (b): the series is

$$
(1 + x + x^2 + \ldots) + (x + x^3 + x^5 + \ldots),
$$
a sum of two geometric series, converging to

$$
t(x) = \frac{1}{1-x} + \frac{x}{1-x^2} = \frac{1+2x}{1-x^2},
$$
for $|x| < 1$. If $|x| > 1$, the terms of the series do not go to $0$ as $n \to \infty$, so it diverges. Thus, its convergence interval is $(-1, 1)$.

10. a) $\frac{x+3}{x^2 - 2} = 1 + \frac{5}{x-2} = 1 - \frac{5}{2-x} = 1 - \frac{5}{2} \frac{1}{1-x/2} = 1 - \frac{5}{2} \sum_{n=0}^{\infty} \frac{x^n}{2^n},$

convergent on $|x| < 2$, i.e. on $(-2, 2)$, as a geometric series.

b) Start with series we know:

$$
\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad \text{(convergent for } |x| < 1 \text{)}
$$

Then $\arctan(x^2) = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \cdots$, also convergent for $|x^3| < 1$, which is the same as $|x| < 1$.

Finally $x \arctan(x^2) = x^3 - \frac{x^7}{3} + \frac{x^{11}}{5} - \cdots$

$$
= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+4},
$$
still convergent for $|x| < 1$, i.e. on $(-1, 1)$.

c) Start with

$$
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
$$

$$
e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots
$$

$$
\int_0^x e^{-x^2} dx = x - \frac{x^3}{3} \frac{x^5}{5} - \frac{x^7}{7} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1) n!}
$$

Constant term is zero since $\int_0^0 e^{-x^2} dx = 0$. Converges for all $x$. 

10. \( (27-x)^{\frac{1}{3}} = 3 \left(1-\frac{x}{27}\right)^{\frac{1}{3}} \)
\[ = 3 \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} \left(-\frac{1}{27}\right)^n x^n \]
(binomial series)
Converges for \( \left| \frac{x}{27} \right| < 1 \), i.e. on \((-27, 27)\)

11. Find derivatives of \( f \):
\[ f(x) = xe^{-x} \]
\[ f'(x) = e^{-x} - xe^{-x} \]
\[ f''(x) = (x-2)e^{-x} \]
\[ f'''(x) = (3-x)e^{-x} \]
\[ f^{(4)}(x) = (x-4)e^{-x} \]
Evaluate at \( x=1 \):
\[ f(1) = 0, \quad f'(1) = -e^{-1}, \quad f''(1) = 2e^{-2}, \quad f^{(4)}(1) = -3e^{-1} \]
The Taylor series is
\[ e^{-x} = \frac{e^{-1}}{2} (x-1)^2 + \frac{2e^{-1}}{6} (x-1)^3 - \frac{3e^{-1}}{24} (x-1)^4 + \ldots \]
(Note: since the \((x-1)^4\) term is zero, the first four terms include the \((x-1)^4\) term.)

12. \[ \sum_{n=0}^{\infty} \frac{4^n}{5^n n!} = \frac{1}{5} \sum_{n=0}^{\infty} \frac{(4/5)^n}{n!} = \frac{1}{5} e^{4/5} \]

13. \[
\begin{array}{cccccccc}
\hline
x & 0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \\
\hline
y & 1 & 1.1 & 1.231 & 1.40254 & 1.62925 & 1.93469 \\
\hline
\end{array}
\]

14. a) \( y' = \frac{x}{y}, \quad y(1) = 2 \)

b) \( yy' = x \quad dy = x \, dx \quad \frac{y^2}{2} = \frac{x^2}{2} + C \),
Setting \( x=1, y=2 \) gives \( 2 = \frac{1}{2} + C \), \( C = \frac{3}{2} \),
so \( \frac{y^2}{2} = \frac{x^2}{2} + \frac{3}{2} \), \( y^2 = x^2 + 3 \)
\[ y = \sqrt{x^2 + 3} \]
15. Let \( a(t) \) be the \% carbon dioxide. Initially, \( a(0) = 0.2 \).
The differential equation is
\[
a'(t) = \frac{1}{100} c(t) - \frac{1}{100} a(t)
\]
flourishing in \( t \), flowing out
with \( t \) in minutes: here \( \frac{1}{100} \) is the vol. per minute divided by the volume of the room. It's both separable and linear.
By either method, the general solution is
\[
a(t) = 0.1 + C e^{-t/100}
\]
The solution with \( a(0) = 0.2 \) is
\[
a(t) = 0.1 + 0.1 e^{-t/100}
\]
16. \( y' + \frac{1}{2x} y = x^{1/2} \). Integrating factor \( A(x) = e^{\int \frac{1}{2x} dx} = e^{\frac{1}{2} \ln x} = \sqrt{x} \).

\[
\int A(x) \, dy = \int 1 \, dx
\]
\[
\sqrt{x} y' + \frac{1}{2 \sqrt{x}} y = 1
\]
\[
\frac{d}{dx} (\sqrt{x} y) = x + C
\]
\[
\sqrt{x} y = x + C
\]

17. \( \int \frac{dy}{1+y} = \int \frac{sin x}{1+cos x} \) ex use \( u = 1 + \cos x \) then \( du = -\sin x \, dx \)
\[
\ln |1+y| = -\ln |1+cos x| + C
\]
\[
1+y = \frac{A}{1+cos x}
\] (where \( A = e^C \) is positive)
\[
y = \frac{A}{1+cos x} - 1
\] (now \( A \) can be negative)

This equation is also linear, but it's much easier to solve using the fact that it is separable.
13. \( \int \frac{dy}{y^2} = \int 2x \, dx \quad - \frac{1}{y} = x^2 + C \quad y = \frac{-1}{x^2 + C} \)

Using \( y(1) = 1 \):

\[
1 = \frac{-1}{1 + C}, \quad 1 + C = -1, \quad C = -2,
\]

\[
y = \frac{-1}{x^2 - 2}
\]