1. $\dfrac{\cos^2(n)}{n^2+1}$ is not decreasing, so the Integral Test does not apply.

2. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos^2(n)}{n^2+1}$ has positive terms, and $\dfrac{\cos^2(n)}{n^2+1} \le \dfrac{1}{n^2+1} < \dfrac{1}{n^2}$, so the series is convergent by comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

3. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n-1)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is the same as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ minus its first term $\frac{1}{1^2} = 1$.

Therefore $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n-1)^2} = \frac{\pi^2}{6} - 1$

4. By the remainder estimate in the integral test, the remainder $\displaystyle\sum_{k=n+1}^{\infty} \frac{1}{k^4}$ after the n^{th} partial sum is less than $\displaystyle\int_n^{\infty} \frac{1}{x^4}\,dx = -\frac{x^{-3}}{3}\Big]_n^{\infty} = \frac{n^{-3}}{3} = \frac{1}{3n^3}$.

We want n large enough to make $\frac{1}{3n^3} < .01$, i.e. $3n^3 > 100$. The smallest such n is $n = 4$.

5. a) Abs. Conv. by limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ (or direct comparison).

b) $\left| \dfrac{n \sin(n)}{n^3+1} \right| \le \dfrac{n}{n^3+1} < \dfrac{1}{n^2}$, hence Abs. Conv. by direct comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

c) Divergent by limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

d) Divergent because the terms $(-1)^n \arctan(n)$ do not $\to 0$ as $n \to \infty$ (note: $\arctan(n) \to \pi/2$ as $n \to \infty$).

e) Abs. Conv. by ratio test.

f) Convergent by alternating series test. The absolute values give a divergent series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n+5}$, by limit comparison with $\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, so the original series is only conditionally convergent, not abs. conv.

g) Equal to $\sum_{x=1}^{\infty} \frac{1}{2^n}$, convergent as a geometric series or by ratio test.

h) Divergent by ratio test.

6. If $p \geq 0$, it's divergent by comparison with the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$. If $p < 0$, the terms are decreasing and we can use the integral test.

$$\int_2^{\infty} \frac{(\ln x)^p}{x}\,dx = \int_{\ln 2}^{\infty} u^p\,du = \frac{u^{p+1}}{p+1}\Big]_{\ln 2}^{\infty}$$

$u = \ln x$
$du = \frac{dx}{x}$

(or $\ln(u)\Big]_{\ln 2}^{\infty}$ if $p = -1$).

The improper integral is divergent if $p \geq -1$, convergent if $p < -1$, so the series is also divergent if $p \geq -1$, convergent if $p < -1$.

7. By the ratio test the series is convergent for $|2(x+1)| < 1$, divergent for $|2(x+1)| > 1$, i.e. convergent on $(-\frac{3}{2}, -\frac{1}{2})$, divergent outside of $[-\frac{3}{2}, -\frac{1}{2}]$.

At $x = -\frac{3}{2}$, we get $\sum_{n=0}^{\infty} \frac{1}{n+1}$, which is divergent. (harmonic series)

At $x = -\frac{1}{2}$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$, which is a convergent alternating series. So the interval of convergence is $(-\frac{3}{2}, -\frac{1}{2}]$, radius of convergence $\frac{1}{2}$.

8. Ratio test gives convergence for $|3x-4|<1$, divergence for $|3x-4|>1$, i.e. convergent on $(1, 5/3)$, divergent outside of $[1, 5/3]$. The radius of convergence is $\frac{1}{3}$.
At $x = 5/3$, we get $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is convergent. At $x=1$, we get $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which is also convergent, since $\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$.
The interval of convergence is $[1, 5/3]$.

9. (a) and (b): the series is

$$(1 + x + x^2 + \cdots) + (x + x^3 + x^5 + \cdots),$$

a sum of two geometric series, converging to

$$f(x) = \frac{1}{1-x} + \frac{x}{1-x^2} = \frac{1+2x}{1-x^2}$$

for $|x|<1$. If $|x|\geq 1$, the terms of the series do not $\to 0$ as $n\to\infty$, so it diverges. Thus the convergence interval is $(-1,1)$.

10. a) $\frac{x+3}{x-2} = 1 + \frac{5}{x-2} = 1 - \frac{5}{2-x} = 1 - \frac{5}{2}\frac{1}{1-x/2} = 1 - \frac{5}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}$,

convergent on $|\frac{x}{2}|<1$, i.e. on $(-2,2)$, as a geometric series.

b) Start with series we know:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad (\text{convergent for } |x|<1)$$

Then $\arctan(x^3) = x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \cdots$, ~~also~~ convergent for $|x^3|<1$, which is the same as $|x|<1$.

Finally $x\arctan(x^3) =$

$$x^4 - \frac{x^{10}}{3} + \frac{x^{16}}{5} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{2n+1},$$

still convergent for $|x|<1$, i.e. on $(-1,1)$.

c) Start with

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

$$\int_0^x e^{-t^2}\,dt = x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)\,n!}.$$

Constant term is zero since $\int_0^0 e^{-t^2}dt = 0$. Converges for all x.

10. d) $(27-x)^{1/3} = 3(1-x/27)^{1/3}$

$$= 3 \cdot \sum_{n=0}^{\infty} \binom{1/3}{n} \frac{(-1)^n x^n}{27^n}.$$

(binomial series)

Converges for $\left|\frac{x}{27}\right| < 1$, i.e. on $(-27, 27)$

11. Find derivatives of f:

$f(x) = xe^{-x}$

$f'(x) = (1-x)e^{-x}$

$f''(x) = (x-2)e^{-x}$

$f'''(x) = (3-x)e^{-x}$

$f^{iv}(x) = (x-4)e^{-x}$

Evaluate at $x=1$: $f(1) = e^{-1}$, $f'(1) = 0$, $f''(1) = -e^{-1}$, $f'''(1) = 2e^{-1}$, $f^{iv}(1) = -3e^{-1}$.

The Taylor series is

$$e^{-1} - \frac{e^{-1}}{2}(x-1)^2 + \frac{2e^{-1}}{6}(x-1)^3 - \frac{3e^{-1}}{24}(x-1)^4 + \cdots$$

[Note: since the $(x-1)^1$ term is zero, the first four terms include the $(x-1)^4$ term.]

12. $$\sum_{n=0}^{\infty} \frac{4^n}{5^{n+1} n!} = \frac{1}{5} \sum_{n=0}^{\infty} \frac{(4/5)^n}{n!} = \frac{1}{5} e^{4/5}$$

13.

x:	0	.1	.2	.3	.4	.5
y:	1	1.1	1.231	1.40254	1.62925	1.93469

14. a) $y' = x/y$, $y(1) = 2$

b) $yy' = x$ $\quad y\,dy = x\,dx \quad \frac{y^2}{2} = \frac{x^2}{2} + C$. Setting $x=1$, $y=2$ gives $2 = \frac{1}{2} + C$, $C = 3/2$, so $\frac{y^2}{2} = \frac{x^2}{2} + 3/2 \quad y^2 = x^2 + 3$

$y = \sqrt{x^2+3}$

15. Let $a(t)$ be the % carbon dioxide. Initially, $a(0) = .2$.

The differential equation is

$$a'(t) = \underbrace{\frac{1}{100}(.1)}_{\text{flowing in}} - \underbrace{\frac{1}{100}a(t)}_{\text{flowing out}}$$

with t in minutes: here $\frac{1}{100}$ is the vol. per minute divided by the volume of the room. It's both seperable and linear.

By either method, the general solution is

$$a(t) = .1 + Ce^{-t/100}.$$

The solution with $a(0) = .2$ is $a(t) = .1 + .1e^{-t/100}$.

16. $y' + \frac{1}{2x}y = \cancel{2}x^{-1/2}$. Integrating factor $A(x) = e^{\int \frac{1}{2x}dx}$

$= e^{\frac{1}{2}\ln(x)} = \sqrt{x}$.

$$\Big\downarrow \cdot A(x)$$

$$\underbrace{\sqrt{x}\,y' + \frac{1}{2\sqrt{x}}y}_{=\,(\sqrt{x}\,y)'} = 1 \quad \xrightarrow{\int} \quad \begin{aligned} \sqrt{x}\,y &= x + C \\ y &= \sqrt{x} + C/\sqrt{x} \end{aligned}$$

17. $\displaystyle\int \frac{dy}{1+y} = \int \frac{\sin x}{1+\cos x}dx$ ← use $u = 1+\cos x$ $\quad du = -\sin x\, dx$

$$\ln|1+y| = -\ln|1+\cos x| + C$$

$$|1+y| = \frac{A}{|1+\cos x|} \qquad (\text{where } A = e^C \text{ is positive})$$

$$1+y = \frac{A}{1+\cos x} \qquad (\text{now } A \text{ can be negative})$$

$$y = \frac{A}{1+\cos x} - 1$$

This equation is also linear, but it's much easier to solve using the fact that it is separable.

18. $\int \frac{dy}{y^2} = \int 2x\,dx \qquad -\frac{1}{y} = x^2 + C \qquad y = \frac{-1}{x^2+C}$

Using $y(1) = 1$: $1 = \frac{-1}{1+C}$, $1 + C = -1$, $C = -2$,

$$y = \frac{-1}{x^2-2}$$