Math 1B—Calculus, Spring 2018—Haiman
Practice problems for Midterm 1

1. Evaluate the integral
$$\int (\ln x)^2 \, dx$$
Answer: $x(\ln x)^2 - 2x \ln x + 2x + C$ (substitute $x = e^u$ and integrate by parts twice).

2. Evaluate the definite integral
$$\int_0^1 \frac{dx}{(x^2+1)^{3/2}}$$
Answer: $1/\sqrt{2}$ (use a trig substitution $x = \tan\theta$).

3. Evaluate the integral
$$\int \frac{x}{2^x} \, dx$$
Answer: $-\frac{2^{-x}(x\ln(2)+1)}{(\ln 2)^2} + C$ (integrate by parts).

4. Evaluate the integral
$$\int \frac{x^3+4x^2+x-1}{x^3+x^2} \, dx$$
Answer: $x + \frac{1}{x} + 2\ln(x) + \ln(x+1) + C$ (use partial fractions).

5. Evaluate the integral
$$\int x \sin^2 x \, dx$$
Answer: $\frac{x^2}{4} - \frac{1}{4} x \sin(2x) - \frac{1}{8}\cos(2x) + C$ (integrate by parts and use a double-angle formula).

6. Evaluate the definite integral
$$\int_0^4 e^{\sqrt{x}} \, dx$$
Answer: $2(1+e^2)$ (substitute $x = u^2$ and integrate by parts).

7. Evaluate the integral
$$\int \sin(3x)\cos(5x) \, dx$$
Answer: $\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x)$ (use angle sum formulas).

8. Evaluate the integral
$$\int \tan^2\theta \sec^4\theta \, d\theta$$

Answer: $(\tan^5\theta)/5 + (\tan^3\theta)/3 + C$ (save a factor $\sec^2\theta\, d\theta$ and express the rest in terms of $\tan\theta$).

9. Evaluate the definite integral

$$\int_0^\pi x \sin x \cos x \, dx$$

Answer: $-\pi/4$ (integrate by parts).

10. Evaluate the integral

$$\int \sqrt{3 + 2x - x^2} \, dx$$

Answer: $\frac{1}{2}(x-1)\sqrt{3+2x-x^2} - 2\sin^{-1}\left(\frac{1-x}{2}\right) + C$ (complete the square and substitute $x = 1 - 2\sin\theta$).

11. Evaluate the integral

$$\int \ln(x^2 - x + 2) \, dx$$

Hint: use integration by parts to get started.

Answer: $\left(x - \frac{1}{2}\right)\ln\left(x^2 - x + 2\right) - 2x + \sqrt{7}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) + C$ (follow the hint, then find $\int \frac{x}{x^2-x+2}\, dx$ by completing the square and using a trig substitution).

12. (a) Use Simpson's rule with $n = 4$ to find a rational number (a fraction with integer numerator and denominator) which approximates the integral

$$\int_1^2 \frac{1}{x} \, dx.$$

Answer: $1747/2520$

(b) Use a calculator to compare your answer with the exact value $\ln(2)$ [this part would not actually be on an exam, since calculators are not allowed].

Answer: $1747/2520 \approx 0.693254$, $\ln(2) \approx 0.693147$

13. Explain why each of the following integrals is improper, and determine whether it is convergent or divergent:

(a)

$$\int_0^\infty \frac{1}{1+x^3} \, dx$$

Answer: Improper because we are integrating on an infinite interval. Convergent by comparison with $\int_1^\infty 1/x^3 \, dx$.

(b)

$$\int_{-1}^0 \frac{1}{1+x^3} \, dx$$

Answer: Improper because the integrand has a discontinuity at $x = -1$. Divergent by factoring $1+x^3 = (1+x)(1-x+x^2)$. The second factor goes to 1 as $x \to -1$, so the function behaves like $1/(1+x)$ near $x = -1$; the integral $\int_{-1}^{0} 1/(1+x)\,dx$ is divergent.

14. Evaluate the improper integral

$$\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx$$

Answer: $1 - e^{-1}$ (evaluate the indefinite integral by substituting $u = 1/x$, then evaluate the improper definite integral by taking a limit).

15. (a) Sketch the region bounded by the x and y axes and the curve

$$y = \frac{1}{\sqrt{x}\,(x+1)}.$$

(b) Find the area of this region, if it is finite. Hint: make a substitution to evaluate the relevant integral.

Answer: $\int_0^\infty \frac{1}{\sqrt{x}\,(x+1)} = \pi$. Substitute $x = u^2$ to evaluate the indefinite integral, then evaluate the improper definite integral by taking a limit. This is a convergent improper integral at both ends of the interval $[0, \infty)$.

16. Find the arc length of the curve

$$y = \frac{x^3}{3} + \frac{1}{4x}$$

from $x = 1$ to $x = 2$.

Answer: $59/24$. The integral is $\int_1^2 \sqrt{1 + (x^2 - 1/(4x^2))^2}\,dx$, which simplifies to $\int_1^2 x^2 + 1/(4x^2)\,dx$.

17. (a) Sketch the curve $y = x^3$ for $0 \le x \le 1$

(b) Sketch the surface obtained by rotating this curve about the x axis.

(c) Find the area of the surface in part (b).

Answer: $(\pi/27)(10^{3/2} - 1)$.

18. A cylindrical tank with a radius of $2\,\mathrm{m}$ is half full of water. Find the hydrostatic force (in Newtons) on one end of the tank. Use the values $\rho = 10^3\,\mathrm{kg/m^3}$ for the density of water and $g = 9.8\,\mathrm{N/kg}$ for the acceleration of gravity.

Answer: $(16/3)\rho g = (16/3)(9.8)(1000)$.

19. Find the centroid of the triangular region with vertices at $(0,0)$, $(3,0)$ and $(0,4)$.

Answer: $(1, 4/3)$.

20. (a) Find a formula for the n-th term a_n (starting at $n = 1$) of the sequence

$$\frac{\sqrt{1\cdot 2}}{1}, \frac{\sqrt{2\cdot 3}}{3}, \frac{\sqrt{3\cdot 4}}{5}, \frac{\sqrt{4\cdot 5}}{7}, \ldots$$

Answer: $a_n = \frac{\sqrt{n(n+1)}}{2n-1}$

(b) Find the limit $\lim_{n\to\infty} a_n$ of this sequence.

Answer: $\lim_{n\to\infty} \frac{\sqrt{n(n+1)}}{2n-1} = \sqrt{\lim_{n\to\infty} \frac{n(n+1)}{(2n-1)^2}} = 1/2.$

21. For each of the following, decide whether the sequence converges, diverges to ∞ or $-\infty$, or diverges without going to $\pm\infty$.

(a) $\lim_{n\to\infty} n\cos(\pi n)$

(b) $\lim_{n\to\infty} \cos(\pi n)/n$

(c) $\lim_{n\to\infty} n^3/2^n$

(d) $\lim_{n\to\infty} \sqrt{n} + \arctan(n^2)$

Answer: (a) Diverges, not to $\pm\infty$. (b) Converges to zero. (c) Converges to zero. (d) Diverges to $+\infty$.

22. Express the repeating decimal $3.272727\ldots$ as a rational number p/q, where p and q are integers with no common factor.

Answer: $36/11$.

23. Find the sum of the geometric series $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \cdots$.

Answer: $2\frac{1}{1-(-2/3)} = 6/5$.