1. Substitute \( u = \sqrt{1+x} \) and integrate by parts to get \( 2 + 2(\sqrt{x} - 1)e^\sqrt{x} \).

2. Save a factor of \( \sec^3 x \) and set \( u = \tan x \) to get \( \frac{1}{3} \tan^3 x + C \).

3. Use \( \sin 3x = \sin x \cos 2x + \sin 2x \cos x \), \( \sin x = \sin(\pi - x) \cos 2x + \sin 2x \cos(x) \) to get \( \frac{1}{3} \tan^3 x + C \), then \( \int_{0}^{\sqrt{2}} = -\frac{1}{3} \).

4. Use \( x = \sec \theta \), get \( \ln |x + \sqrt{x^2 - 25}| + C \).

5. Complete the square: \( x - x^2 = \frac{1}{4} - (x - \frac{1}{2})^2 \), then use \( x = \frac{1}{2} + \frac{1}{2} \sin \theta \), answer \( \frac{\sqrt{2}}{8} \). For \( \cos \theta \) recognize the \( \int \) as giving the area of a semicircle with radius \( \frac{1}{2} \).

6. Use \( x = \tan \theta \), get \( \frac{x}{\sqrt{x^2 + 1}} + C \).

7. Use partial fractions \( \frac{x}{x^2 + 1} = \frac{1}{3} \frac{x + 1}{x^2 - x + 1} - \frac{1}{3} \frac{1}{x + 1} \), complete the square \( x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4} \), and integrate to get \( \frac{\pi}{3} - \ln(2) \).

8. Midpoint \( M_2 = \frac{B}{4} \), Trapezoidal \( T_2 = \frac{1}{4} \), Simpson \( S_4 = \frac{1}{3} T_2 + \frac{2}{3} M_2 = \frac{2\sqrt{3} + 1}{12} \). \( M_2 \approx 0.483 \) is over, \( T_2 = 0.25 \) is under, \( S_4 \approx 0.372 \) is the closest to \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.392 \). The graph of \( \sqrt{x^2 - x^2} \) is concave downward; this is why \( T_2 \) is under and \( M_2 \) is over. In \( S_4 \), these errors tend to cancel.

9. \( \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} \) has \( \lim_{x \to \infty} \tan(x) = \frac{\pi}{2} \), \( \lim_{x \to -\infty} \tan(x) = -\frac{\pi}{2} \), so \( \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} \) converges to \( \tan^{-1}(x) \bigg|_{-\infty}^{\infty} = \pi \).

10. The integral is improper because \( \tan x \) has a discontinuity at \( \frac{\pi}{2} \). To see if it converges, we need to evaluate \( \lim_{a \to \frac{\pi}{2}^-} \int_{0}^{a} \tan x \, dx \) and \( \lim_{a \to \frac{\pi}{2}^+} \int_{a}^{\pi} \tan x \, dx \). Since \( \int_{0}^{\pi} \tan x \, dx = \ln|\sec x| \to -\infty \) as \( x \to \frac{\pi}{2} \), the limits don't exist, and the integral diverges.
11. Find \( ds = \sqrt{y'^2 + 1} \) \( dx = \sqrt{9x^2 + 1} \), then find \( \int_0^4 \frac{dx}{\sqrt{9x^2 + 1}} = \frac{8}{27} \left( 10 \sqrt{10} - 1 \right) \).

12. The ellipse is \( \frac{x^2}{4} + y^2 = 1 \).

For \( y = \sqrt{1 - x^2/4} \), the area element \( dA = 2ny \, ds \) is \( \frac{n}{2} \sqrt{16 - 3x^2} \).

Find \( \int_{-2}^{2} \frac{n}{2} \sqrt{16 - 3x^2} \, dx = 2\pi + \frac{8n^2}{3\sqrt{3}} \) using trig sub \( x = \frac{4}{\sqrt{3}} \sin \theta \), or in integral table.

13. Taking coordinate \( y = 0 \) at bottom of dam, the width of the dam at height \( y \) is \( 15 + \frac{y}{2} \), and the water depth is \( 8 - y \), giving force \( \int_{0}^{8} \rho g (8-y)(15+\frac{y}{2}) \, dy \)

\[ = \frac{1568 \times 9800}{3} \, N \]

14. By symmetry, \( \bar{x} = 0 \).

The area is \( A = \int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin x \bigg|_{-\pi/2}^{\pi/2} = 2 \).

To find \( \bar{y} \), can either use

\[ \bar{y} = \frac{1}{A} \int_{0}^{1} y \cdot 2 \cos^2 y \, dy \quad \text{or} \quad \bar{y} = \frac{1}{A} \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{2} \, dx \],

getting \( \bar{y} = \frac{\pi}{8} \). (For the first method, integrate by parts; for the 2nd, use half-angle formula \( \cos^2 x = \frac{1 + \cos 2x}{2} \).)

15. \( \frac{1}{x} = \frac{1}{2 - (2-x)} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{n!} \),

by geometric series. The radius of convergence is 2.

16. Substitute \(-x^2\) for \( x \) in the binomial series \( (1+x)^{1/2} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \)

to get \( 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} + \ldots \).
17. Divergent because $1 + \frac{1}{n^2}$ does not go to 0 as $n \to \infty$.

18. The absolute value series

$$\sum_{n=2}^{\infty} \frac{1}{n(n^2+1)}$$

converges by the integral test: $\int_{2}^{\infty} \frac{1}{x(x^2+1)} \, dx \leq \int_{2}^{\infty} \frac{1}{u^2} \, du$.

19. Convergent by all series tests, but not absolutely convergent, by $\delta$ test applied to $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$. So, conditionally convergent.

20. Use the ratio test:

$$\lim_{n \to \infty} \frac{(n+1)!^2}{(2n+2)!} \cdot \frac{1}{(n!)^2} = \lim_{n \to \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1 \Rightarrow \text{convergent}.$$  

The series is positive, so it's absolutely convergent.

21. It's a geometric series $\frac{1}{x^2+4} = \frac{1}{1+(x/2)^2}$, convergent for $1 \left| x/2 \right| < 1$, i.e., $\left| x \right| < 2$, so the radius of convergence is 2.

22. By the $\delta$ test, the remainder term $R_5 = \frac{1}{36} + \frac{1}{48} + \ldots$ is less than $\int_{5}^{\infty} \frac{1}{x^2} \, dx = -\frac{1}{x \mid_{5}^{\infty}} = \frac{1}{5}$.

23. a) $f(x)$ is continuous because $\lim_{x \to 0} \frac{e^{x^2}-1}{x^2} = 1$ (by L'Hopital, or by definition of $y'_{(0)}$ for $y = e^{x^2}$).

b) $e^x = (x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots)$

Divide by $x$ to get $f(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$

e) The radius of convergence is 0 because the series for $e^x$ converges for all $x$. 
25. Factor: \( y' = (x+1)(y+1) \)

\[
\frac{dy}{y+1} = (x+1) \, dx \quad \Rightarrow \quad \ln |y+1| = \frac{x^2}{2} + x + C
\]

\[ y = A e^{\frac{x^2}{2} + x} - 1. \quad y(0) = 0 \Rightarrow A = 1, \]

\[ y = e^{\frac{x^2}{2} + x} - 1. \]

26. \( y' - (x+1) y = x+1. \) Integrating factor \( e \int -(x+1) \, dx = e^{\frac{-x^2}{2} + x} \)

\[ \left( e^{\frac{-x^2}{2} + x} y \right)' = e^{\frac{-x^2}{2} + x} (x+1). \]

Integrate both sides, using \( u = \frac{x^2}{2} + x, \quad du = (x+1) \, dx \) on the right:

\[ e^{\frac{-x^2}{2} + x} y = e^{\frac{-x^2}{2} + x} + C. \]

\[ y = Ce^{\frac{-x^2}{2} + x} - 1, \quad y(0) = 0 \Rightarrow C = 1, \quad y = e^{\frac{-x^2}{2} + x} - 1. \]

27. \( r^2 + 4r + 4 = (r+2)^2 = 0 \) has double root \( r = -2, \) so

\[ y = (A + Bx)e^{-2x}. \quad y(0) = A = 1 \]

\[ y(0) = B - 2A = -2 \Rightarrow B = 0 \]

\[ y = e^{-2x}. \]

28. For homogeneous eqn \( y'' + 5y' + 6y = 0, \quad r^2 + 5r + 6 = (r+2)(r+3) = 0 \)

has roots \( r = -2, -3, \) giving \( y_c = Ae^{-2x} + C e^{-3x}. \)

Since \( e^{-2x} \) is a solution of the complementary equation, find \( y_p \)

Using \( y_p = Ae^{2x} + B xe^{-2x} \)

\[ y'_p = 2Ae^{2x} + B (1-2x) e^{-2x} \]

\[ y''_p = 4Ae^{2x} + 4B (x-1) e^{-2x} \]

\[ y'' + 5y' + 6y = 20Ae^{2x} + Be^{-2x} = e^{2x} + e^{-2x} \]

\[ \Rightarrow A = \frac{1}{20}, \quad B = 1, \]

\[ y = y_p + y_c = \frac{1}{20} e^{2x} + xe^{-2x} + Ce^{-2x} + C e^{-3x} \]
29. For the homogenous eqn. \( y'' + 2y' + 2y = 0 \), \( r = -2 \pm \sqrt{4 - 4} = -1 \pm i \),
so \( y_e = e^{-x} (A \cos x + B \sin x) \). Take \( y_p = G \) then
\[ y'' + 2y' + 2y = 2C = 0 \Rightarrow C = 1, \text{ so } y = y_p + y_e = 1 + e^{-x} (A \cos x + B \sin x). \]
\( y(0) = 1 + A = 0 \Rightarrow A = -1 \)
\( y(\frac{\pi}{2}) = 1 + e^{-\frac{\pi}{2}} B = 0 \Rightarrow B = e^{-\frac{\pi}{2}} \)
y = \frac{1}{1 - e^{-x}} (\cos x + e^{\frac{\pi}{2}} \sin x).

30. a) Period of oscillation is \( 2\pi \sqrt{\frac{m}{k}} = \frac{2\pi}{10} \approx 0.628 \) sec
b) With \( m = 2 \times 10^9, c = 2 \times 10^5, k = 2 \times 10^6 \),
\( my'' + cy' + ky = 0 \) has \( c^2 - 4km < 0 \), so the solution has exponentially decaying oscillations.

31. Plugging in \( y = \sum_{n=0}^{\infty} C_n x^n \), \( y' = \sum_{n=0}^{\infty} (n+1) C_{n+1} x^n \), \( xy' = \sum_{n=0}^{\infty} n C_n x^n \),
y'' = \( \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n \)
\[ \sum_{n=0}^{\infty} \left( (n+2)(n+1) C_{n+2} + n C_n - (n+1) C_{n+1} + C_n \right) x^n = 0 \]
\[ \sum_{n=0}^{\infty} \left( (n+2)(n+1) C_{n+2} - (n+1) C_{n+1} + C_n \right) x^n = 0 \]
so \( C_{n+2} = \frac{C_{n+1} - C_n}{n+2} \) for all \( n = 0, 1, \ldots \)

Initial conditions give \( C_0 = y(0) = 1, C_1 = y'(0) = 0 \).

Then \( (n=0) \) \( C_2 = \frac{C_1 - C_0}{2} = \frac{1}{2} \)
\( (n=1) \) \( C_3 = \frac{C_2 - C_1}{3} = \frac{1}{6} \)
\( (n=2) \) \( C_4 = \frac{C_3 - C_2}{4} = \frac{1}{12} \)
y = \( 1 - \frac{1}{2} x^2 - \frac{1}{6} x^3 + \frac{1}{12} x^4 + \cdots \).
32. a) Plug in \[ y = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy' = \sum_{n=0}^{\infty} n a_n x^n \]

\[
\sum_{n=0}^{\infty} (n+1) a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

Equate coefficients: \[ (n+1) a_n = \frac{1}{n!} \quad \Rightarrow \quad a_n = \frac{1}{(n+1)!} \]

\[ y = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n! (n+1)} \]

is the only series solution (this is the same function as in problem 23).

b) The equation \[ y' + y = e^{x/x} \] is linear, with integrating factor \[ e^{\int_{1/x}^{1} dx} = x \], giving back the original equation

\[ xy' + y = e^x \]

\[ (xy)' \]

Integrate both sides to get

\[ xy = e^x + C \]

\[ y = \frac{e^x + C}{x} \]

This general solution has an undetermined constant, as expected.

But it is discontinuous at \( x=0 \) unless \( C=-1 \), in which case \( f(x) = \frac{e^x - 1}{x} \) extends to a continuous function with \( f(0) = 1 \).

This \( f(x) \) is the only possible Maclaurin series solution, since such a solution can't be discontinuous at \( x=0 \).