1. [8 pts each] Decide whether each of the following series is divergent, absolutely convergent, or conditionally convergent, and give reasons for your answers.

(a)

$$\sum_{n=0}^{\infty}(-1)^n \frac{\sqrt{n}+1}{n^2+1}$$

Absolutely convergent. Use limit comparison of the absolute values $\sum_{n=0}^{\infty} \frac{\sqrt{n}+1}{n^2+1}$ with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

(b)

$$\sum_{n=1}^{\infty} n^3 2^{-n}$$

Absolutely convergent. Use ratio test or root test.

(c)

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

Divergent. Use comparison with harmonic series, or compare with $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$, which is divergent by $\int$ test. In principle you could use the $\int$ test directly, but $\int \frac{1}{\ln x}\,dx$ can't be expressed in closed form.

2. [15 pts] Find a power series representation for the function

$$f(x) = \int_0^x \sin(t^2)\, dt$$

and determine its interval of convergence.

Start with $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Then $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

Integrating gives

$$f(x) = \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}.$$

The constant term is 0 because $f(0) = 0$.

It converges for all x because the series we started with does.

Name ______________________ Student ID______________________

3. [10 pts] If a function $J(x)$ is defined by the power series

$$J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2},$$

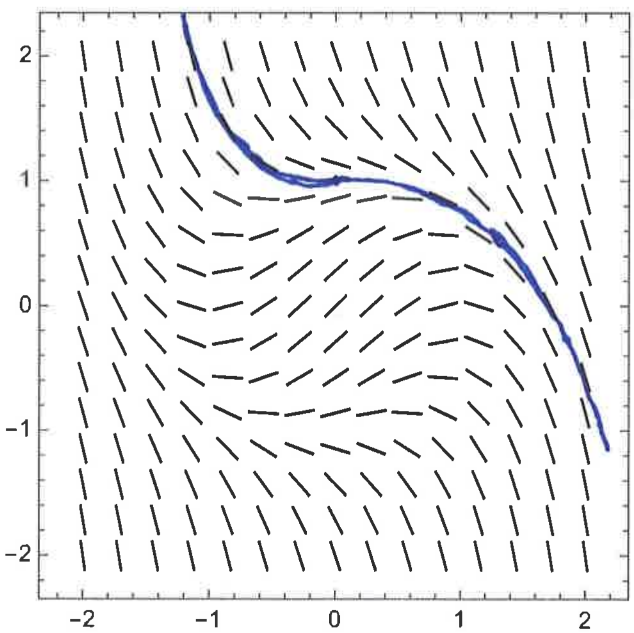
find the value $J^{(6)}(0)$ of the sixth derivative of $J(x)$ at $x = 0$.

The x^6 term (for $n=3$) is $-\frac{1}{36}x^6$.

Therefore $\frac{J^{(6)}(0)}{6!} = -\frac{1}{36}$, $J^{(6)}(0) = -\frac{6!}{36} = -\frac{720}{36} = -20.$

4. (a) [8 pts] The direction field plotted below corresponds to one of the differential equations listed at left. Circle the correct equation and explain your choice.

(b) [8 pts] On the direction field plot, sketch the graph of the solution $y = f(x)$ of the chosen equation with initial condition $y(0) = 1$.



(1) $y' = \sin(x)$

(2) $y' = (x+1)(y+1)$

(3) $y' = 1 - x^2 - y^2$

(4) $y' = 1 - \arctan(y)$

The plot doesn't match the direction fields for (1), (2) or (4) because, for instance, (1) should have slope $y'=0$ on the y axis ($x=0$), (4) should have slope $y'=1$ on the x axis ($y=0$), (2) should have positive slope $y'>0$ throughout the first quadrant ($x>0$, $y>0$).

5. [15 pts] Find the general solution of the differential equation

$$y' + 2\frac{y}{x} = \cos(x^3).$$

Linear. Multiply by integrating factor $A(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$ to get

$(x^2y)' = x^2y' + 2xy = x^2\cos(x^3).$

Integrate: $x^2y = \int x^2\cos(x^3)\,dx = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C$

$(u = x^3,\ du = 3x^2dx) \quad = \frac{1}{3}\sin(x^3) + C.$

Solution: $y = \frac{1}{3}\frac{\sin(x^3)}{x^2} + C\cdot\frac{1}{x^2}$

6. (a) [15 pts] Solve the initial value problem

$$(x+1)y' = x/y, \quad y(0) = 1.$$

Separable, $yy' = \frac{x}{x+1} = 1 - \frac{1}{x+1}$

$\int y\,dy = \int 1 - \frac{1}{x+1}\,dx$, $\quad \frac{y^2}{2} = x - \ln|x+1| + C.$

Set $x=0$, $y=1$ to get $\frac{1}{2} = C$, so

$\frac{y^2}{2} = x - \ln|x+1| + \frac{1}{2}$ $\qquad y^2 = 2x - 2\ln|x+1| + 1$

$y = \sqrt{2x - 2\ln(x+1) + 1}$

see part (b)

(b) [5 pts] What is the domain of the solution $y = f(x)$? (Remember that the *domain* means the set of x values for which the solution is valid.)

The general solution has a discontinuity at $x = -1$. This means that solutions with initial condition $y(x_0) = y_0$ have domain $(-1, \infty)$ if $x_0 > -1$, or $(-\infty, -1)$ if $x_0 < -1$. The solution with $y(0) = 1$ therefore has domain $(-1, \infty)$. This justifies dropping the absolute value signs in $\ln|x+1|$, since $x+1$ is positive on the domain of the solution.