Name Solutions Student ID

Section time & instructor

Math 1B—Calculus, Spring 2018—Haiman
Midterm Exam 1

Instructions:

- Wait until time to start is announced before looking at the exam questions.

- Write your name, ID number and section time and instructor's name on this page now. Before turning in your exam, write your name and ID number at the top of the first side of the second page.

- Write your answers on the exam paper in the space provided. Do preliminary work on scratch paper, then write a clear and concise answer giving your solution and enough steps to justify it.

  Points may be deducted for incorrect or irrelevant parts of a solution even if a correct answer is included.

  If you need more space for your solution to a problem, attach an extra page and write your name and ID number on it. Extra pages should not usually be needed.

- You may use one sheet (written on both sides) of prepared notes. No other notes, books, calculators, or other electronic devices are allowed.

- There are 6 questions, worth a total of 100 points.

1. [16 pts] Evaluate the integral

$$\int x^{19} \ln(x)\, dx$$

Integrate by parts with

$$u = \ln(x) \qquad dv = x^{19}\, dx$$

$$du = \frac{1}{x}\, dx \qquad v = \frac{1}{20} x^{20}$$

$$uv - \int v\, du = \frac{1}{20} x^{20} \ln(x) - \frac{1}{20} \int x^{19}\, dx$$

$$= \frac{1}{20} x^{20} \ln(x) - \frac{1}{400} x^{20} + C$$

2. [16 pts] Evaluate the definite integral

$$\int_0^{\pi} \sin^3(x) \cos^2(x)\, dx$$

Save a sin(x) factor :

$$\int_0^{\pi} \sin^2(x) \cos^2(x) \cdot \sin(x)\, dx$$

Substitute $u = \cos(x)$ and use $\sin^2(x) = 1 - \cos^2(x)$:

$$-\int_1^{-1} (1-u^2)\, u^2\, du = \int_{-1}^{1} u^2 - u^4\, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} \Big]_{-1}^{1} = \frac{1}{3} - \frac{1}{5} - \left(\frac{-1}{3} - \frac{-1}{5} \right) = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$$

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3. [16 pts] If the improper integral converges, evaluate it. If it diverges, explain why.

$$\int_{-\pi/2}^{\pi/2} \tan x \, dx$$

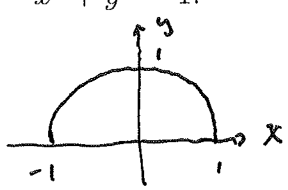
It is improper at both ends, so we must take limits of $\int_a^b \tan x \, dx$ as $b \to \pi/2^-$ and $a \to (-\pi/2)^+$ separately.

Now $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$, so
$(u = \cos x)$

$\int_a^b \tan x \, dx = -\ln|\cos b| + \ln|\cos a|$.

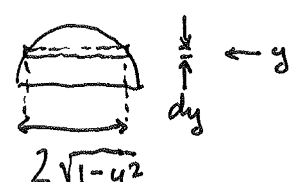
As $b \to (\pi/2)^-$, $\cos b \to 0^+$, $\ln|\cos b| \to -\infty$, so this integral diverges. (It also diverges at the other end, but divergence at either end is enough to say it diverges.)

4. [16 pts] Find the centroid of the region between the x axis and the upper half of the circle $x^2 + y^2 = 1$.



By symmetry, $\bar{x} = 0$.
There are two ways to find $\bar{y}$.

Method 1: $\bar{y} = \frac{1}{A} \int y \, dA$

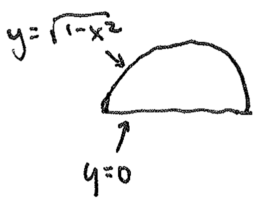


$dA = 2\sqrt{1-y^2}\, dy$

$= \frac{1}{A} \int_0^1 2y \sqrt{1-y^2} \, dy$

$= -\frac{2}{\pi} \int_1^0 u^{1/2} \, du = \frac{2}{\pi} \int_0^1 u^{1/2} \, du = \frac{4}{3\pi} u^{3/2} \Big|_0^1 = \frac{4}{3\pi}$ $\quad (u = 1 - y^2)$

Method 2: $\bar{y} = \frac{1}{A} \int \frac{1}{2}(f(x)^2 - g(x)^2)\, dx = \frac{2}{\pi} \int_{-1}^1 \frac{1}{2}(1 - x^2)\, dx$



$= \frac{2}{\pi}\left(\frac{x}{2} - \frac{x^3}{6}\right)\Big|_{-1}^1 = \frac{2}{\pi}\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right) = \frac{4}{3\pi}$

Answer $(\bar{x}, \bar{y}) = \left(0, \frac{4}{3\pi}\right)$

5. (a) [10 pts] Write down an integral which gives the arc length of the curve $y = 4x^{3/2}$ between the origin and the point $(1, 4)$.

$$y' = 6x^{1/2} \qquad ds = \sqrt{1+(y')^2}\, dx = \sqrt{1+36x}\, dx$$

$$S = \int_0^1 \sqrt{1+36x}\, dx$$

(b) [10 pts] Evaluate the integral in part (a).

Substitute $u = 1+36x$, $\quad dx = \frac{du}{36}$

$$\int_0^1 \sqrt{1+36x}\, dx = \frac{1}{36}\int_1^{37} u^{1/2}\, du = \frac{1}{36}\cdot \frac{2}{3}u^{3/2}\Big]_1^{37}$$

$$= \frac{1}{54}\left(37^{3/2}-1\right) = \frac{1}{54}\left(37\sqrt{37}-1\right)$$

6. [16 pts] If the series is convergent, find the sum. If it is divergent, explain why.

$$\sum_{n=1}^{\infty} \frac{2 \cdot 3^n + (-1)^n}{5^n}$$

It's the sum of two geometric series (both of which converge)

$$\sum_{n=1}^{\infty} 2\cdot\left(\frac{3}{5}\right)^n + \sum_{n=1}^{\infty}\left(\frac{-1}{5}\right)^n$$

$$= \frac{6/5}{1-3/5} + \frac{-1/5}{1-(-1/5)} = 3 - \frac{1}{6} = \frac{17}{6}$$