Name Solutions Student ID

Section time & instructor

Math 1B—Calculus, Spring 2018—Haiman
Final Exam

Instructions:

- Wait until time to start is announced before looking at the questions.

- Write your name, ID number and section time and instructor's name on this page now.

  Before turning in your exam, write your name and ID number in the space provided on the first side of each remaining page.

- Write your answers on the exam paper in the space provided. Do preliminary work on scratch paper, then write a clear and concise answer giving your solution and enough steps to justify it.

  Points may be deducted for incorrect or irrelevant parts of a solution even if a correct answer is included.

  If you need more space for your solution to a problem, attach an extra page and write your name and ID number on it. Extra pages should not usually be needed.

- You may use two sheets (written on both sides) of prepared notes. No other notes, books, calculators, or other electronic devices are allowed.

- There are 14 questions, worth a total of 100 points.

1. [7 pts] Evaluate the indefinite integral.

$$\int \sec^3(x)\tan(x)\,dx$$

$= \int \sec^2(x) \cdot \sec(x)\tan(x)\,dx$

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$\int u^2\,du = \frac{u^3}{3} + C$

$= \frac{\sec^3(x)}{3} + C$

2. [7 pts] Evaluate the indefinite integral.

$$\int \frac{x^3+1}{x^2-4}\,dx$$

$\frac{x^3+1}{x^2-4} = x + \frac{4x+1}{x^2-4} = x + \frac{4x+1}{(x+2)(x-2)} = x + \frac{7/4}{x+2} + \frac{9/4}{x-2}$

$\int x + \frac{7/4}{x+2} + \frac{9/4}{x-2}\,dx = \frac{x^2}{2} + \frac{7}{4}\ln|x+2| + \frac{9}{4}\ln|x-2| + C$

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3. [7 pts] If we use the Midpoint Rule with two subdivisions to approximate

$$\ln(2) = \int_1^2 \frac{1}{x}\,dx,$$

we get $M_2 = \frac{24}{35}$. If we use the Trapezoidal Rule instead, we get $T_2 = \frac{17}{24}$. What result do we get if we use Simpson's Rule with four subdivisions? Express your answer as a fraction in lowest terms.

$$S_4 = \frac{1}{3}T_2 + \frac{2}{3}M_2 = \frac{17}{72} + \frac{48}{105} = \frac{1747}{2520}$$

$$\frac{48}{105} = \frac{16}{35}$$

4. [7 pts] Determine whether the improper integral converges and justify your answer. You are not required to evaluate the integral.

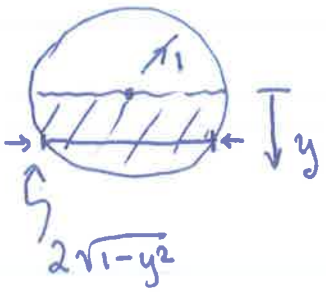
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

It converges if both $\int_0^\infty e^{-x^2}dx$ and $\int_{-\infty}^0 e^{-x^2}dx$ converge.

We can see that $\int_0^\infty e^{-x^2}dx$ converges by comparison with the convergent integral $\int_0^\infty e^{-x}dx = -e^{-x}\Big]_0^\infty = 1$, since $0 < e^{-x^2} < e^{-x}$ for $x > 1$. We can see that $\int_{-\infty}^0 e^{-x^2}dx$ converges by comparison with $\int_{-\infty}^0 e^x\,dx = e^x\Big]_{-\infty}^0 = 1$, since $e^{-x^2} < e^x$ for $x < -1$. Or note that since $\int_0^\infty e^{-x^2}dx$ converges, so does $\int_{-\infty}^0 e^{-x^2}dx$, by symmetry.

5. [7 pts] A cylindrical milk tank lies horizontally, so its circular ends are vertical. The tank is 1 meter in radius and 5 meters long. Find the force due to hydrostatic pressure on one end of the tank when it is half full of milk.

For simplicity, take the weight per unit volume of milk to be 10000 Newtons per cubic meter.



$$F = \int_0^1 \rho g \, 2\sqrt{1-y^2} \, y \, dy \qquad \rho g = 10000 \text{ N}$$

$$\int_0^1 2\sqrt{1-y^2} \, y \, dy \qquad u = 1-y^2 \quad du = -2y \, dy$$

$$= -\int_1^0 u^{1/2} \, du = -\frac{2}{3} u^{3/2} \Big]_1^0 = \frac{2}{3} u^{3/2} \Big]_0^1 = \frac{2}{3}$$

$$F = \frac{2}{3} \cdot 10000 \text{ N}$$

6. [7 pts] Find a simple algebraic expression for the function $f(x)$ whose Taylor series about $x = 3$ is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1/2)(-3/2)\cdots(1/2-n)}{n!}(x-3)^n.$$

This is the binomial series for

$$((x-3)+1)^{-1/2} = \frac{1}{\sqrt{x-2}}.$$

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7. [7 pts] Determine whether the series is divergent, conditionally convergent, or absolutely convergent. Justify your answer.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log_2 n}$$

It's convergent by alternating series test. For absolute convergence, use the $\int$ test on $\sum_{n=2}^{\infty} \frac{1}{n \log_2 n} = \sum_{n=2}^{\infty} \frac{\ln(2)}{n \ln(n)}$.

$\ln(2) \int_2^{\infty} \frac{1}{x \ln x}\, dx = \ln(2) \int_{\ln(2)}^{\infty} \frac{1}{u}\, du$ is divergent, so $\sum_{n=2}^{\infty} \frac{1}{n \log_2 n}$ is

$u = \ln x$

$du = \frac{1}{x} dx$

divergent. Therefore $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log_2 n}$ is conditionally convergent.

8. [7 pts] Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n(n+1)}{2^n}(x+3)^n$$

Use ratio test:

$$\lim_{n\to\infty} \frac{(n+1)(n+2)}{2^{n+1}} \frac{2^n}{n(n+1)} \frac{(x+3)^{n+1}}{(x+3)^n}$$

$$= \lim_{n\to\infty} \frac{(n+2)(x+3)}{2n} = \frac{x+3}{2}.$$

The interval of convergence is $\left|\frac{x+3}{2}\right| < 1$, i.e. $|x+3| < 2$, or $x \in (-5, -1)$, plus possibly the endpoints.

At $x = -1$ we get $\sum_{n=0}^{\infty} n(n+1)$. At $x = -5$ we get $\sum_{n=0}^{\infty} (-1)^n n(n+1)$.

Both are divergent since ~~their~~ terms do not $\to 0$ as $n \to \infty$.

So the interval of convergence is $(-5, -1)$.

9. [7 pts] Use Euler's method with step size 1/2 to approximate the solution of

$$y' = \frac{x}{y} + 1, \quad y(0) = 1$$

at $x = 0$, $x = 1/2$, and $x = 1$.

$(x_0, y_0) = (0, 1)$

$(x_1, y_1) = (\frac{1}{2}, \frac{3}{2})$

$(x_2, y_2) = (1, 13/6)$

$y_1 = y_0 + \frac{1}{2}\left(\frac{x_0}{y_0} + 1\right) = y_0 + \frac{1}{2} = \frac{3}{2}$

$y_2 = y_1 + \frac{1}{2}\left(\frac{x_1}{y_1} + 1\right) = y_1 + \frac{1}{2} \cdot \frac{4}{3} = \frac{3}{2} + \frac{2}{3}$

$= \frac{13}{6}$

10. [7 pts] Find the general solution of the differential equation.

$$y' = y/x + \ln x$$

$y' - y/x = \ln x$ is linear. Use integrating factor $e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

to get

$\frac{y'}{x} - \frac{y}{x^2} = \frac{\ln x}{x}$ $\xrightarrow{\int}$ $\frac{y}{x} = \int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} = \frac{1}{2}\ln^2 x + C$

$\quad \| $

$(y/x)'$

$u = \ln x, \; du = \frac{1}{x}\,dx$

$y = \frac{1}{2} x \ln^2(x) + Cx$

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11. [7 pts] Solve the initial value problem.

$$y'' + 4y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 0$$

$r^2 + 4r + 5 = 0$ has roots $r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$, so

$y = e^{-2x}(C_1 \cos x + C_2 \sin x)$

$y' = e^{-2x}((C_2 - 2C_1)\cos x - (C_1 + 2C_2)\sin x)$

$y(0) = C_1 = 1$

$y'(0) = C_2 - 2C_1 = 0 \Rightarrow C_2 = 2$

$y = e^{-2x}(\cos x + 2 \sin x)$

12. [7 pts] Find the general solution of the differential equation.

$$y'' - 3y' + 2y = e^x + e^{-x}$$

$r^2 - 3r + 2 = (r-1)(r-2) = 0$ has roots $r = 1, 2$, so $y_c = C_1 e^x + C_2 e^{2x}$. Since e^x is a solution of the complementary homogeneous equation, we use a trial solution $y = Axe^x + Be^{-x}$ to find y_p. Then:

$y = Axe^x + Be^{-x}$

$y' = A(x+1)e^x - Be^{-x}$

$y'' = A(x+2)e^x + Be^{-x}$

$y'' - 3y' + 2y = -Ae^x + 6Be^{-x} = e^x + e^{-x}$

$\Rightarrow A = -1,\ B = 1/6$. $y_p = -xe^x + \frac{e^{-x}}{6}$

$y = y_p + y_c = -xe^x + \frac{e^{-x}}{6} + C_1 e^x + C_2 e^{2x}$

13. [7 pts] Solve the boundary value problem.

$$y'' + 2y' + y = 0, \quad y(-1) = e,\ y(1) = 0$$

$r^2 + 2r + 1 = (r+1)^2 = 0$ has double root $r = -1$, giving

$y = e^{-x}(A + Bx)$.

$y(-1) = e(A-B) = e \Rightarrow A - B = 1$

$y(1) = e^{-1}(A+B) = 0 \Rightarrow A + B = 0$

$\Rightarrow A = \frac{1}{2},\ B = -\frac{1}{2}$

$$y = \frac{1}{2}e^{-x}(1-x)$$

14. [5 pts] (a) If $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ is a solution of the differential equation

$$y'' - xy' - 2y = \frac{1}{1-x},$$

find an expression for c_{n+2} in terms of c_n, valid for all $n = 0, 1, \ldots$

$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n \qquad xy' = \sum_{n=0}^{\infty} n c_n x^n \qquad y = \sum_{n=0}^{\infty} c_n x^n$

$$y'' - xy' - 2y = \sum_{n=0}^{\infty} \Big((n+1)(n+2)c_{n+2} \underbrace{- nc_n - 2c_n}_{} \Big) x^n = \sum_{n=0}^{\infty} 1\, x^n$$

$= (n+2)\big((n+1)c_{n+2} - c_n\big)$

geometric series $\frac{1}{1-x}$

Equating coefficients,

$$(n+2)\big((n+1)c_{n+2} - c_n\big) = 1$$

$$(n+1)c_{n+2} - c_n = \frac{1}{n+2}$$

$$c_{n+2} = \frac{c_n + \frac{1}{n+2}}{n+1} = \frac{(n+2)c_n + 1}{(n+1)(n+2)}$$

(b) [4 pts] Find the first four non-zero terms of the solution which satisfies the initial conditions $y(0) = 1$, $y'(0) = 0$.

$c_0 = 1,\quad c_1 = 0,\quad c_2 = \frac{c_0 + \frac{1}{2}}{1} = \frac{3}{2},\quad c_3 = \frac{c_1 + \frac{1}{3}}{2} = \frac{0 + \frac{1}{3}}{2} = \frac{1}{6},$

$c_4 = \frac{c_2 + \frac{1}{4}}{3} = \frac{7/4}{3} = \frac{7}{12}$

$$y = 1 + \frac{3}{2}x^2 + \frac{1}{6}x^3 + \frac{7}{12}x^4 + \cdots$$