1. (8 pts each part) (a) Express

$$\lim_{x \to 3} \frac{2^x - 8}{x - 3}$$

as the derivative $f'(a)$ of some function $f(x)$ at some number a.

$$\lim_{x \to 3} \frac{2^x - 8}{x - 3} = f'(3) \quad \text{for} \quad f(x) = 2^x$$

(b) Evaluate it.

$$f'(x) = 2^x \ln(2)$$

$$f'(3) = 8 \ln(2)$$

2. (14 pts) Differentiate $\tan^{-1}(\sqrt{4x+1})$.

$$\frac{1}{(\sqrt{4x+1})^2 + 1} \cdot \frac{1}{2\sqrt{4x+1}} \cdot 4 = \frac{1}{(2x+1)\sqrt{4x+1}}$$

3. (14 pts) Differentiate $(x+1)^{x/2}$.

with $f(x) = (x+1)^{x/2}$, $\quad \ln f(x) = \frac{x}{2} \ln(x+1)$

$$(\ln f(x))' = \frac{1}{2} \ln(x+1) + \frac{x}{2(x+1)}$$

$$f'(x) = f(x)\,(\ln f(x))' = (x+1)^{x/2}\left(\frac{1}{2}\ln(x+1) + \frac{x}{2(x+1)}\right)$$

4. (14 pts) Find the point (x, y) where the graph of

$$y = xe^{-x}$$

has a horizontal tangent line.

We want $y' = 0$.

$y' = e^{-x} + x e^{-x}(-1) = (1-x)e^{-x}$,

so $y' = 0 \Rightarrow x = 1, \quad y = 1e^{-1} = \frac{1}{e}$

5. (14 pts) If $e^{xy} = y$, find an expression for dy/dx in terms of x and y.

$$e^{xy} = y$$

$$\Downarrow$$

$$e^{xy}(y + xy') = y'$$

$$(1 - xe^{xy})y' = ye^{xy}$$

$$y' = \frac{ye^{xy}}{1 - xe^{xy}}, \text{ or } \frac{y}{e^{-xy} - x},$$

$$\text{or } \frac{y^2}{1 - xy}, \text{ or } \frac{y}{1/y - x}$$

(these are all equal)

6. (14 pts) The pressure P, volume V and temperature T of a gas are related by

$$PV = KT,$$

where K is a constant. If the pressure, volume and temperature are functions of time t, find dT/dt when $P = 50$ N/cm^2, $V = 1000$ cm^3, $T = 300$ K, $dP/dt = 10$ N/cm^{2}s and $dV/dt = -50$ cm^3/s.

Since $50 \cdot 1000 = k \cdot 300$, $\quad k = \frac{500}{3}$.

Now $P\frac{dV}{dt} + V\frac{dP}{dt} = k\frac{dT}{dt}$

$$\underbrace{50(-50) + 1000(10)}_{=\,7500} = \frac{500}{3}\frac{dT}{dt}$$

$$\frac{dT}{dt} = \frac{7500 \cdot 3}{500} = 15 \cdot 3 = 45 \ K/s$$

7. (14 pts) Use a linear approximation or differentials to approximate $\sqrt[4]{80}$. Hint: $3^4 = 81$.

Let $f(x) = x^{1/4}$, $\quad a = 81$.

Then $f(a) = 3$

$$f'(x) = \frac{1}{4}x^{-3/4}$$

$$f'(a) = \frac{1}{4 \cdot 3^3} = \frac{1}{4 \cdot 27}$$

$$\sqrt[4]{80} \approx f(81) + f'(81)(80-81)$$

$$= 3 + \frac{1}{4 \cdot 27}(-1) = 3 - \frac{1}{4 \cdot 27} = 3 - \frac{1}{108} = \frac{323}{108}$$