1. (a) (5 pts) Find the value of the constant A that makes the function

$$f(x) = \begin{cases} x^2 + Ax & x < 2 \\ \sqrt{x} & x \geq 2 \end{cases}$$

continuous for all x.

Need limits as $x \to 2^+$, $x \to 2^-$ to match:

$$2^2 + 2 \cdot A = \sqrt{2}$$

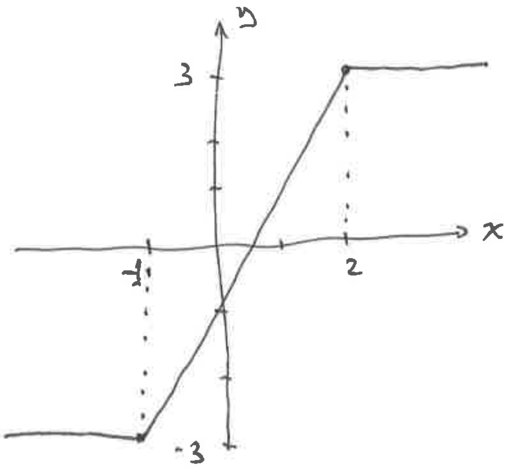
$$2A = \sqrt{2} - 4$$

$$A = \frac{\sqrt{2}}{2} - 2$$

2. (5 pts) Simplify $x^{2/\ln x}$

$$x^{2/\ln x} = e^{\ln x \cdot (2/\ln x)} = e^2$$

3. (6 pts) For what values of x is the function $f(x) = |x+1| - |x-2|$ differentiable? Sketch the graph of the function and explain how your answer is related to the graph.



Differentiable at all x except $x = -1, 2$. At these points, the change in slope means that the limits from the left and right in the definition of the derivative do not agree.

Name ______________________ Student ID______________________

4. (5 pts) Evaluate the limit, possibly as an infinite limit, or explain why it does not exist.

$$\lim_{x\to 0} \frac{x - \ln(x+1)}{1 - \cos(x)}$$

Use L'Hospital (0/0 form twice).

First time gives $\lim_{x\to 0} \frac{1 - \frac{1}{x+1}}{\sin x}$.

Second time gives $\lim_{x\to 0} \frac{(x+1)^{-2}}{\cos x} = \frac{1}{1} = 1$ by direct substitution.

5. (5 pts) Evaluate the limit, possibly as an infinite limit, or explain why it does not exist.

$$\lim_{x\to \pi} \frac{\sin x}{x}$$

The denominator does not go to zero, so the quotient is continuous at π. Thus

$$\lim_{x\to\pi} \frac{\sin x}{x} = \frac{0}{\pi} = 0.$$

6. (5 pts) Evaluate the limit, possibly as an infinite limit, or explain why it does not exist.

$$\lim_{x\to\infty} (\ln(x) - x)$$

The limit is $-\infty$ because x grows faster than $\ln(x)$.

More precisely, $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$ (e.g., by L'Hospital ∞/∞ form).

Then $\lim_{x\to\infty} (\ln(x) - x) = \lim_{x\to\infty} x\left(\frac{\ln(x)}{x} - 1\right) = -\infty$, since

$\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty}\left(\frac{\ln(x)}{x} - 1\right) = -1.$

7. (6 pts) Differentiate the function $f(x) = \sqrt{1+\sqrt{1+\sqrt{x}}}$.

$$f'(x) = \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2} \cdot \frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2} \cdot \frac{1}{2}x^{-1/2}$$

$$= \frac{1}{8\sqrt{1+\sqrt{1+\sqrt{x}}}\ \sqrt{1+\sqrt{x}}\ \sqrt{x}}$$

8. (6 pts) Find the maximum and minimum values of the function $f(x) = x + 4/x$ on the interval $[1, 3]$.

$f'(x) = 1 - \frac{4}{x^2}$. For critical points, $1 - \frac{4}{x^2} = 0 \iff \frac{4}{x^2} = 1$

$\iff x^2 = 4$

$\iff x = \pm 2$.

Of the points $x = \pm 2$, only $x = 2$ is in the interval. Evaluting $f(x)$ at endpoints and c.p.'s:

$f(1) = 5$

$f(2) = 4$

$f(3) = \frac{13}{3}$

Since $4 < \frac{13}{3} < 5$, the minimum is $f(2) = 4$, and the maximum is $f(1) = 5$.

9. (6 pts) If $y\ln(y) = x$, find $y'(x)$ and $y''(x)$ at the point $(0, 1)$.

$y' \ln y + y \frac{1}{y} y' = 1 \quad \Rightarrow \quad y'(\ln y + 1) = 1$. At $y = 1$, this gives $y' = 1$.

Differentiate implicitly a second time to get

$$y''(\ln y + 1) + y' \frac{1}{y} y' = 0.$$

Since we already know $y' = 1$ when $y = 1$, this gives

$$y''(1) + 1 = 0,$$

$$y'' = -1.$$

Name ______________________ Student ID______________________

10. (6 pts) The pressure P, volume V, and temperature T of a gas are related by $PV = 20T$. All three quantities vary with time. Find T and T' when $P = 40$, $V = 25$, $P' = 5$, and $V' = -2$.

For T, $20T = 40 \cdot 25 = 1000$, so $T = 50$

For T', $P'V + PV' = 20T'$

$$= 5 \cdot 25 - 40 \cdot 2 = 125 - 80 = 45$$

$$T' = \frac{45}{20} = \frac{9}{4} = 2.25$$

11. (a) (3 pts) Without using any calculations, give a reason why there must be a point (a, b) on the graph $y = x^3$, with $0 < a < 1$, where the tangent line to the graph is parallel to its secant line through the points $(0, 0)$ and $(1, 1)$.

This is the Mean Value Theorem for the function $f(x) = x^3$ (which is continuous and differentiable for all x) on the interval $[0, 1]$.

Note that (a, b) in the statement of this problem are the coordinates of a point, not the endpoints of an interval.

(b) (3 pts) Calculate the coordinates of the point (a, b) in the first part of the question.

Need y' equal to the slope of the secant line, i.e. $y' = 1$.

So $y' = 3x^2 = 1$, $x^2 = \frac{1}{3}$, $a = x = \frac{1}{\sqrt{3}}$, $b = a^3 = \frac{1}{3\sqrt{3}}$

$$(a, b) = \left(\frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}}\right).$$

12. (6 pts) A particle moves along the x-axis with velocity $v(t) = t^2 - 1$. It starts at time $t = 0$ at the origin, $x(0) = 0$. Find its position $x(2)$ at time $t = 2$, and the total distance travelled during the time interval $[0, 2]$. Note: the distance travelled is not necessarily the same as the final position.

$$x(2) = x(0) + \int_0^2 t^2 - 1 \, dt = 0 + \frac{t^3}{3} - t \Big|_0^2 = \frac{2}{3}.$$

Distance travelled is

$$\int_0^2 |t^2 - 1| \, dt = \int_0^1 1 - t^2 \, dt + \int_1^2 t^2 - 1 \, dt$$

$$= t - \frac{t^3}{3}\Big]_0^1 + \frac{t^3}{3} - t\Big]_1^2 = \frac{2}{3} + \left(\frac{2}{3} - \left(-\frac{2}{3}\right)\right) = 2.$$

13. (6 pts) By using Riemann sums with three terms to approximate the integral $\int_1^2 \frac{1}{x} dx$, find numbers B and C such that $B \leq \ln(2) \leq C$. The numbers B and C should be expressed as fractions with integer numerator and denominator.

Since $1/x$ is decreasing, the lower sum B uses right endpoints:

$$B = \frac{1}{3}\left(\frac{3}{4} + \frac{3}{5} + \frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{15 + 12 + 10}{60} = 37/60$$

$1 \quad 4/3 \quad 5/3 \quad 2$

$f(1) = 1 \quad f(4/3) = 3/4 \quad f(5/3) = 3/5 \quad f(2) = 1/2$

with $f(x) = 1/x$.

The upper sum C uses left endpoints:

$$C = \frac{1}{3}\left(1 + \frac{3}{4} + \frac{2}{5}\right) = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{20 + 15 + 12}{60} = \frac{47}{60}.$$

Name ______________________ Student ID______________________

14. (5 pts) Evaluate the definite integral

$$\int_1^e \frac{\ln(x)}{x}\, dx$$

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$$\int_0^1 u\, du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}.$$

15. (5 pts) Evaluate the indefinite integral

$$\int x\sqrt{x+2}\, dx$$

$u = x+2 \qquad du = dx$

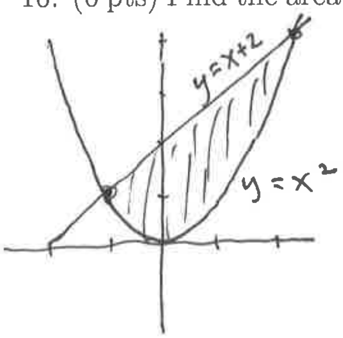
$$\int (u-2)\,u^{1/2}\, du = \int u^{3/2} - 2u^{1/2}\, du$$

$$= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2}$$

$\left(= \frac{2}{15}(x+2)^{3/2}(3x-4),\ \text{if you prefer} \right)$

16. (6 pts) Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x+2$.



The two graphs meet where $x^2 = x+2$,

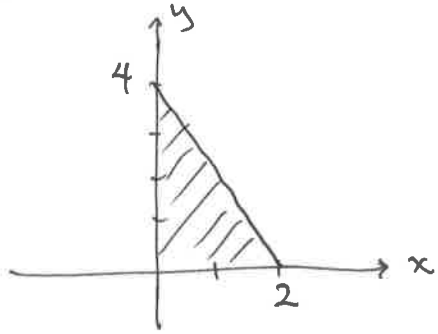
$x^2 - x - 2 = (x-2)(x+1) = 0, \quad x = -1, 2.$

The area is then given by

$$\int_{-1}^{2} x+2-x^2\, dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big]_{-1}^{2} = \frac{10}{3} - \left(-\frac{7}{6}\right)$$

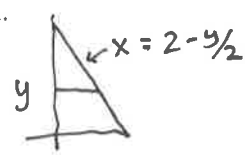
$$= \frac{27}{6} = 9/2$$

17. (a) (2 pts) Draw an accurate sketch of a region in the plane such that the solid of revolution obtained by rotating this region about the y axis is a cone with height 4 and circular base of radius 2.



(b) (4 pts) Set up and evaluate an integral which gives the volume of this cone, using your choice of method.

By disks:

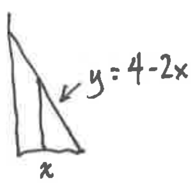


$$\int_0^4 \pi(2-y/2)^2\,dy = -2\pi\int_2^0 u^2\,du = 2\pi\int_0^2 u^2\,du$$

$u = 2-y/2$, $dy = -2\,du$

$$= 2\pi \frac{u^3}{3}\Big]_0^2 = 16\pi/3$$

By shells:



$$\int_0^2 2\pi x(4-2x)\,dx = 2\pi\int_0^2 4x-2x^2\,dx$$

$$= 2\pi\left(2x^2-\frac{2x^3}{3}\right)\Big|_0^2 = 2\pi\left(8-\frac{16}{3}\right) = 2\pi\frac{8}{3} = 16\pi/3$$

18. (5 pts) Find the number a such that the average value of the function $f(x) = x^2$ on the interval $[-a, a]$ is equal to 1.

The average is $\bar{f} = \frac{1}{2a}\int_{-a}^{a} x^2\,dx = \frac{1}{2a}\frac{x^3}{3}\Big]_{-a}^{a} = \frac{1}{2a}\frac{2a^3}{3} = \frac{a^2}{3}$.

Setting $\bar{f} = 1$ gives $a = \sqrt{3}$ (we want the positive square root since the interval goes from $-a$ to a).