1. In this problem, $f(x) = xe^{-x}$.

(a) (8 pts) Find the intervals where f is increasing or decreasing, and all local maxima and minima of f.

$f'(x) = (1-x)e^{-x}$

On $(-\infty, 1)$: $f' > 0$, f increasing

On $(1, \infty)$: $f' < 0$, f decreasing

Crit point at $x=1$ is a local (and absolute) maximum, with $f(1) = e^{-1}$

(b) (8 pts) Find the intervals where the graph of f is concave upward or downward, and find all inflection points on the graph.

$f''(x) = (x-2)e^{-x}$

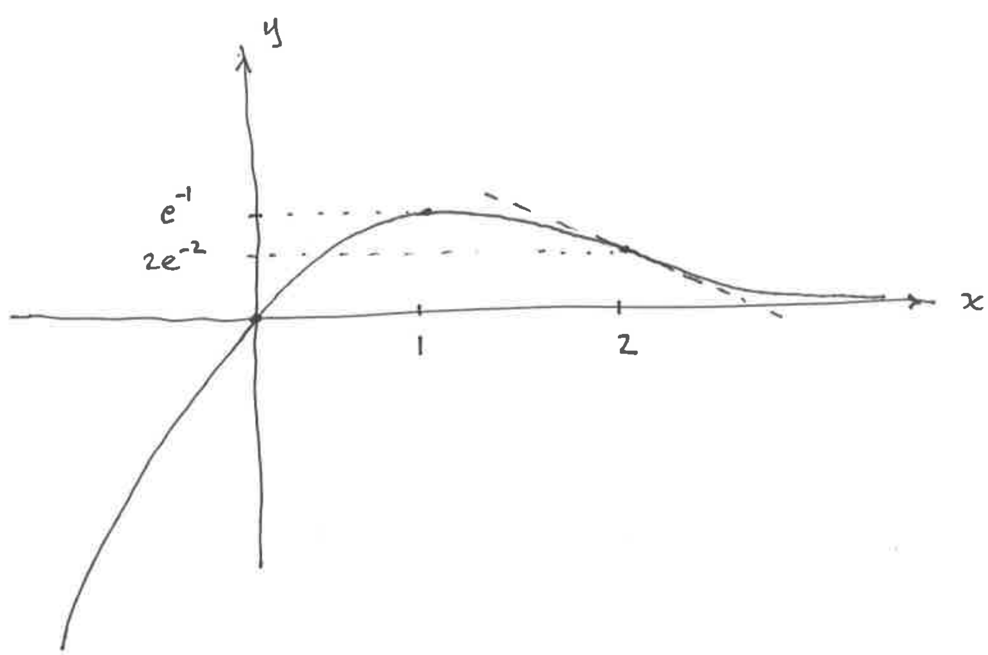
On $(-\infty, 2)$: $f'' < 0$, concave down

On $(2, \infty)$: $f'' > 0$, concave up

Inflection point at $x=2$, $f(2) = 2e^{-2}$

(c) (8 pts) Sketch the graph of f, accurately displaying the information found in parts (a) and (b). Your sketch should also be accurate at $x = 0$ and as $x \to \pm\infty$.

Besides (a) and (b) we have $f(0) = 0$, $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = 0$ (so the $+x$ axis is a horizontal asymptote).



Name ______________________ Student ID______________________

2. (15 pts) Compute the indefinite integral

$$\int (x - \frac{1}{x})\sqrt{x}\, dx.$$

$\int (x - \frac{1}{x})\sqrt{x}\, dx = \int x^{3/2} - x^{-1/2}\, dx = \frac{2}{5}x^{5/2} - 2x^{1/2} + C$

(or $(\frac{2}{5}x^2 - 2)\sqrt{x} + C$)

3. (8 pts each) Find the two limits:

(a)

$$\lim_{x\to\pi/2} \frac{\sin 2x}{\cos x}$$

By L'Hospital (type $\frac{0}{0}$), $\lim_{x\to\pi/2} \frac{\sin 2x}{\cos x} = \lim_{x\to\pi/2} \frac{2\cos 2x}{-\sin x} = \frac{2(-1)}{(-1)} = 2$

(b)

$$\lim_{x\to 0} \frac{\sin 2x}{\cos x}$$

Don't use L'Hospital! $\lim_{x\to 0} \frac{\sin 2x}{\cos x} = \frac{0}{1} = 0$

4. (15 pts) Compute the definite integral

$$\int_{-1}^{1} \frac{1}{x^2+1}\, dx.$$

$\int_{-1}^{1} \frac{1}{x^2+1}\, dx = \tan^{-1} x \Big]_{-1}^{1} = \pi/4 - (-\pi/4) = \pi/2$

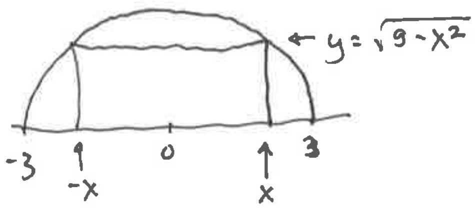
5. (15 pts) If $g(x)$ is defined by $g(x) = \int_1^{3x} \sin(t^2)\, dt$, find $g'(x)$.

$g(x) = h(3x)$, where $h(x) = \int_1^x \sin(t^2)\, dt$.

Then $h'(x) = \sin(x^2)$ by Fund. Theorem of Calculus;

$g'(x) = 3h'(3x) = 3\sin((3x)^2) = 3\sin(9x^2)$.

6. (15 pts) Find the rectangle of largest area which has its bottom side on the x-axis and its two top corners on a circle of radius 3 centered at the origin.



In terms of x, the area is $A(x) = 2x\sqrt{9-x^2}$. We are to maximize it for x in $[0,3]$. To find critical values,

$$A'(x) = 2\sqrt{9-x^2} + 2x\,\frac{1}{2}\,\frac{1}{\sqrt{9-x^2}}\cdot -2x$$

$$= \frac{2(2x^2-9)}{\sqrt{9-x^2}}.$$

$$= 0 \quad \text{when} \quad x = 3/\sqrt{2}$$

Since $A(0) = 0$ and $A(3) = 0$, the maximum is $A(3/\sqrt{2}) = 9$.

Specifically, the largest rectangle has vertical sides at $x = \pm 3/\sqrt{2}$ and top side at $y = \sqrt{9-(3/\sqrt{2})^2} = \sqrt{9/2} = 3/\sqrt{2}$, with area $2\frac{3}{\sqrt{2}}\cdot\frac{3}{\sqrt{2}} = 9$.