Name Solutions Student ID

Section time & instructor

Math 1A—Calculus, Spring 2017—Haiman
Midterm Exam 1

Instructions:

- Wait until time to start is announced before looking at the exam questions.

- Write your name, ID number and section time and instructor's name on this page. After the exam starts, put your name and ID number on the second page.

- Write your answers on the exam paper in the space provided. Do preliminary work on scratch paper, then write a clear and concise answer giving your solution and enough steps to justify it.

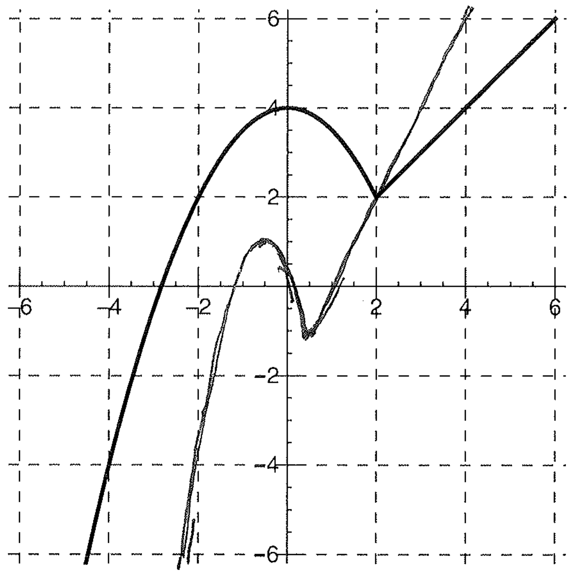
  Points may be deducted even from correct answers if they are cluttered with extraneous material.

  If you need more space for an answer, attach an extra page, and write your name and ID number on it. Ordinarily, extra pages should not be needed.

- You may use one sheet (written on both sides) of prepared notes. No other notes, books, calculators, or other electronic devices are allowed.

- There are 7 questions, for a total of 100 points.

1. (12 pts) The graph of a function $y = f(x)$ is shown. On the same grid, sketch the graph of the function $y = f(2x+1) - 3$. Make your sketch accurate enough to show clearly how it is related to the graph of f.



Move the graph 1 unit to the left, then shrink it horizontally by a factor of 2, then move it 3 units down, to get the new graph.

2. (10 pts) Simplify $8^{\log_2(3)+\log_4(5)}$.

$8^{\log_2(3)} = 2^{3\log_2(3)} = 3^3 = 27$

$8^{\log_4(5)} = 2^{3\cdot\frac{1}{2}\log_2(5)} = 5^{3/2} = 5\sqrt{5}$

$8^{\log_2(3)+\log_4(5)} = 8^{\log_2(3)}\, 8^{\log_4(5)} = 27\cdot 5\sqrt{5} = 135\sqrt{5}$

3. (12 pts) Find the value of the constant C that makes the ~~function~~ following function continuous for all x.

$$f(x) = \begin{cases} Cx - 3, & x < 1, \\ e^{\sqrt{x}}, & x \geq 1 \end{cases}$$

It's continuous for $x \neq 1$ for any value of C. To make it continuous at $x=1$, we need $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x)$.

Now, $\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(Cx-3) = C\cdot1-3 = C-3$

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} e^{\sqrt{x}} = e^{\sqrt{1}} = e.$

Therefore, $C-3=e$, $C=e+3$.

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4. (6 pts each) (a) Find all vertical asymptotes of the graph of the function

$$y = \frac{1}{x^2(x-1)}$$

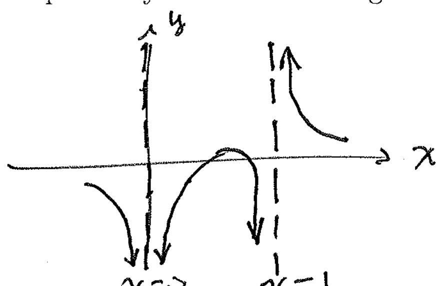
There are two vertical asymptotes : the line $x=0$ and the line $x=1$.

(b) Give the values of any limits relevant to finding the vertical asymptotes.

$\lim\limits_{x\to 0} \frac{1}{x^2(x-1)} = -\infty$ since $x^2 \to 0$ through positive values, and $x-1 \to -1$.

$\lim\limits_{x\to 1^+} \frac{1}{x^2(x-1)} = +\infty$, $\lim\limits_{x\to 1^-} = -\infty$, since $x^2 \to 1$ and $x-1 \to 0$ through positive values as $x \to 1^+$, or through negative values as $x \to 1^-$.

(c) Draw a sketch showing how the graph looks near each vertical asymptote. Your graph does not have to be precisely accurate as long as the behavior near the asymptote or asymptotes is clear.



5. (12 pts) What can you conclude about the solution of the equation

$$\cos(x) = x,$$

given that $\cos(0) = 1$, $\cos(\pi/6) = \sqrt{3}/2 = 0.866\ldots$, $\cos(\pi/4) = \sqrt{2}/2 = 0.707\ldots$, $cos(\pi/3) = 1/2$, $\cos(\pi/2) = 0$, and $\pi = 3.14\ldots$. Locate the solution as accurately as you can based on the given information. Specify any theorems and any properties of functions needed to justify your answer.

The function $f(x) = \cos(x) - x$ is continuous with the following values :

x	$\cos(x) - x$
0	1
$\pi/6$	$\sqrt{3}/2 - \pi/6 \approx 0.87 - 0.52 > 0$
$\pi/4$	$\sqrt{2}/2 - \pi/4 \approx 0.71 - 0.78 < 0$
$\pi/3$	$1/2 - \pi/3 \approx .5 - 1.04 < 0$
$\pi/2$	$0 - \pi/2 \approx 0 - 1.57 < 0$

A solution of the equation is where $f(x) = 0$. By IVT, there is a solution in the interval $(\pi/6, \pi/4)$.

6. (6 pts each) (a) Find the domain and range of the function $f(x) = \ln(1 - \sqrt{x})$.

The domain is $[0, 1)$ because we must have $x \geq 0$ for $\sqrt{x}$ to be defined, and $1 - \sqrt{x} > 0$, thus $\sqrt{x} < 1$ and $x < 1$, for $\ln(1 - \sqrt{x})$ to be defined. The range is $(-\infty, 0]$, the logs of numbers $1 - \sqrt{x}$ in $(0, 1]$.

(b) Give a reason why f is one-to-one.

It is decreasing.

(c) Find the inverse function g of f. What are the domain and range of g?

Solve $y = \ln(1 - \sqrt{x})$ for x: $e^y = 1 - \sqrt{x}$, $1 - e^y = \sqrt{x}$, $x = (1 - e^y)^2$. So $g(y) = (1 - e^y)^2$, or $g(x) = (1 - e^x)^2$.

The domain of g is $(-\infty, 0]$, the range of f. The range of g is $[0, 1)$, the domain of f.

7. (6 pts each) (a) Show that if $\epsilon > 0$ and $4 < x < 4 + 4\epsilon$, then $2 < \sqrt{x} < 2 + \epsilon$. Algebra hint: $4 + 4\epsilon < 4 + 4\epsilon + \epsilon^2 = (2 + \epsilon)^2$.

By the hint, $4 < x < 4 + 4\varepsilon < 4 + 4\varepsilon + \varepsilon^2 = (2+\varepsilon)^2$

Since $\sqrt{x}$ is an increasing function we can take square roots in $4 < x < (2+\varepsilon)^2$ to get $2 < \sqrt{x} < 2 + \varepsilon$.

(Technically, $\sqrt{(2+\varepsilon)^2}$ should be $|2+\varepsilon|$, but $|2+\varepsilon| = 2+\varepsilon$ since $\varepsilon > 0$ implies $2 + \varepsilon > 0$.)

(b) Part (a) verifies the value of some limit, either $\lim_{x\to a} f(x) = L$ or $\lim_{x\to a^+} f(x) = L$ or $\lim_{x\to a^-} f(x) = L$. What limit is it? Express your answer as one of these three forms of limits with the appropriate values filled in for $f(x)$, a and L.

$$\lim_{x \to 4^+} \sqrt{x} = 2$$

(for this limit we should really show $4 < x < 4 + 4\varepsilon \Rightarrow 2 - \varepsilon < \sqrt{x} < 2 + \varepsilon$, but of course $2 < \sqrt{x}$ implies $2 - \varepsilon < \sqrt{x}$, since $\varepsilon > 0$).

(c) What value of δ (in terms of ϵ) is used in part (a) to establish the value of the limit?

$\delta = 4\varepsilon$.