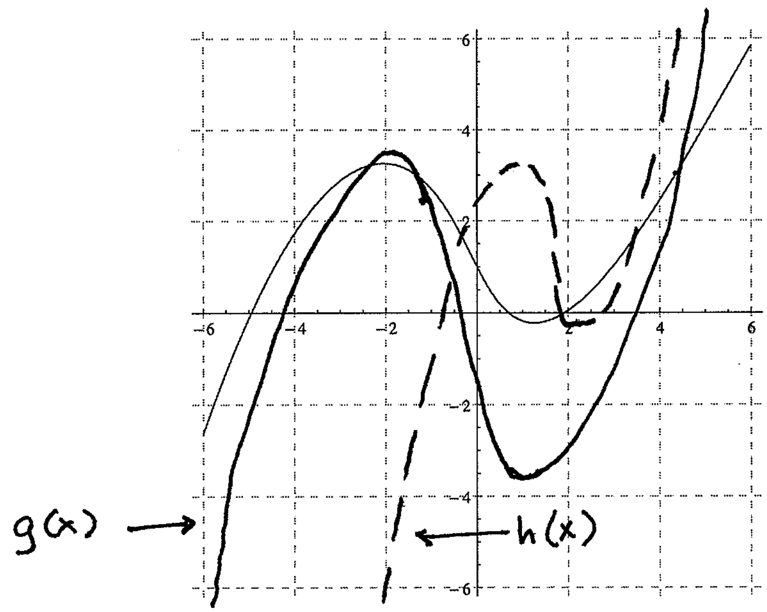
1. (6 pts) The graph of a function $f(x)$ is shown. On the same grid, sketch the graphs of $g(x) = 2f(x) - 3$ and $h(x) = f(2x - 3)$. Label your two graphs so we can tell which is which.



2. (6 pts) Evaluate the limit if it exists, either as a number or an infinite limit.

$$\lim_{x \to \infty} 5\sqrt{x} - 3x$$

$$\lim_{x \to \infty} 5\sqrt{x} - 3x = \lim_{x \to \infty} -3x\left(1 - \frac{5}{3\sqrt{x}}\right) = \lim_{x \to \infty} -3x = -\infty$$

3. (6 pts) Evaluate the limit if it exists, either as a number or an infinite limit.

$$\lim_{x \to \pi/2} \ln(2x/\pi) \sec x$$

$$\lim_{x \to \pi/2} \ln(2x/\pi) \sec x = \lim_{x \to \pi/2} \frac{\ln(2x/\pi)}{\cos x}$$

$$\underset{(\frac{0}{0}\text{ L'Hospital})}{=} \lim_{x \to \pi/2} \frac{1/x}{-\sin x} \underset{(\text{direct sub})}{=} \frac{2/\pi}{-1} = -2/\pi$$

4. (6 pts) Find the equation of the tangent line to the curve $x^3 + y^3 = 9(x - y)$ at the point $(2, 1)$.

Differentiate implicitly: $3x^2 + 3y^2 y' = 9 - 9y'$

$(3y^2+9)y' = 9 - 3x^2 \quad y' = \frac{9-3x^2}{3y^2+9} = \frac{-3}{12} = -\frac{1}{4}$ at $(2,1)$.

The tangent line is therefore

$y - 1 = -\frac{1}{4}(x-2)$, or $y = -\frac{1}{4}x + \frac{3}{2}$.

5. (6 pts) Find the maximum and minimum values of the function $f(x) = x^4 - 8x^2$ on the interval $[-1, 3]$.

$f'(x) = 4x^3 - 16x = 4x(x^2-4) = 4x(x-2)(x+2)$.

C.P.'s in $[-1, 3]$ are $x = 0$, $x = 2$. Evaluating $f(x)$ at C.P.'s and endpoints:

$f(-1) = -7$

$f(0) = 0$

$f(2) = -16$

$f(3) = 81 - 8 \cdot 9 = 9$

Minimum: -16

Maximum: 9

6. (6 pts) Differentiate the function $f(x)$ defined by

$$f(x) = \int_0^{x^2} \sec(u)\, du.$$

You do not need to evaluate the integral to solve this problem.

Let $g(x) = \int_0^x \sec(u)\, du$. Then $g'(x) = \sec(x)$ by Fundamental Theorem of Calculus, and $f(x) = g(x^2)$, so $f'(x) = g'(x^2) \cdot 2x = 2x \sec(x^2)$ by Chain Rule.

7. (a) (4 pts) Show that Newton's method for computing a sequence of approximations $x_1, x_2, \ldots$ to $\sqrt{a}$ leads to the formula

$$x_{n+1} = \frac{x_n + a/x_n}{2}$$

$\sqrt{a}$ is root of $f(x) = x^2 - a = 0$. Newton gives $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^2 - a}{2x_n} = \frac{x_n}{2} + \frac{a}{2x_n} = \frac{x_n + a/x_n}{2}$$

(b) (2 pts) Find the second and third approximations to $\sqrt{2}$ given by this scheme when the first approximation is $x_1 = 1$.

$$x_2 = \frac{1+2}{2} = \frac{3}{2}, \qquad x_3 = \frac{3/2 + 4/3}{2} = \frac{17}{12}$$

8. (6 pts) Find the first and second derivatives $f'(x)$ and $f''(x)$ of the function

$$f(x) = \cos(\ln x)$$

$$f'(x) = -\sin(\ln x)/x$$

$$f''(x) = -\cos(\ln x)/x^2 + \sin(\ln x)/x^2$$

$$= \frac{\sin(\ln x) - \cos(\ln x)}{x^2}$$

9. (6 pts) If you know that $-1 \leq f'(x) \leq 1$ on the interval $[0, 2]$, and $f(0) = 8$, what can you conclude about the value of $f(2)$, and why?

By MVT, $\frac{f(2) - f(0)}{2}$ is a value $f'(c)$ for some c in $[0,2]$,

so $-1 \leq \frac{f(2)-f(0)}{2} \leq 1$. Therefore $f(0) - 2 \leq f(2) \leq f(0) + 2$,

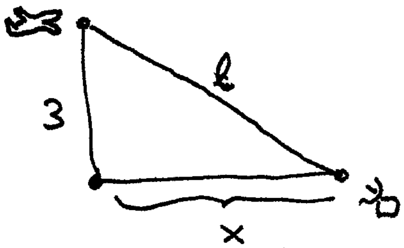
$6 \leq f(2) \leq 10$.

10. (7 pts) For the function $f(x) = 1/x^2$, find the point c in the interval $[1, 4]$ where $f(c)$ is equal to the average value of $f(x)$ on $[1, 4]$.

The average is $\bar{y} = \frac{1}{3}\int_1^4 \frac{1}{x^2}\,dx = \frac{1}{3}\left(\frac{-1}{x}\right)\Big]_1^4 = \frac{1}{3}\left(1 - \frac{1}{4}\right)$

$= \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$. We have $f(c) = \frac{1}{4}$ for $c = 2$.

11. (6 pts) An airplane is flying horizontally, due north, at an altitude of 3 miles and a speed of 500 miles per hour. A radar station is on the ground ahead. At what rate is the distance between the plane and the radar station decreasing when the plane is 4 miles south of the station?



$$\ell^2 = x^2 + 9$$

$$2\ell \frac{d\ell}{dt} = 2x \frac{dx}{dt}.$$

When $x = 4$, $\ell = 5$, and we are given $\frac{dx}{dt} = -500$, so $\frac{d\ell}{dt} = \frac{x}{\ell}\frac{dx}{dt} = \frac{4}{5}(-500) = -400.$

So ℓ is decreasing at a rate of 400 miles per hour.

12. (6 pts) Evaluate the indefinite integral

$$\int \frac{x^2}{\sqrt{3x^3+2}}\, dx$$

Substitute $u = 3x^3 + 2$

$du = 9x^2\, dx$

$$\int \frac{x^2}{\sqrt{3x^3+2}}\, dx = \frac{1}{9}\int u^{-1/2}\, du = \frac{2}{9} u^{1/2} + C = \frac{2}{9}\sqrt{3x^3+2} + C$$

13. (6 pts) Evaluate $\int_0^5 f(x)\, dx$, where

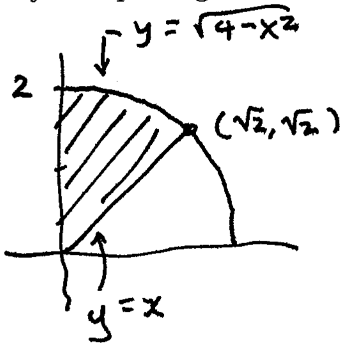
$$f(x) = \begin{cases} x^2 & x < 1 \\ 1/x & x \geq 1 \end{cases}$$

$$\int_0^1 x^2\, dx + \int_1^5 \frac{1}{x}\, dx = \left.\frac{x^3}{3}\right]_0^1 + \ln x\Big]_1^5 = \frac{1}{3} + \ln 5$$

14. (7 pts) Evaluate the definite integral

$$\int_0^{\sqrt{2}} \sqrt{4-x^2} - x \, dx$$

by interpreting it as an area.



The integral gives the shaded area, which is $\frac{1}{8}$ of a circle of radius 2, so equal to

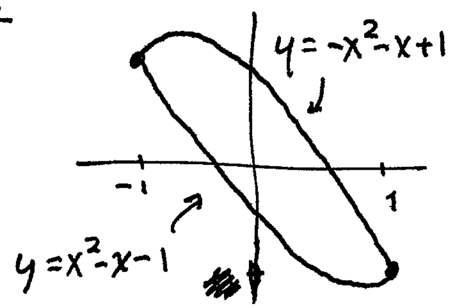
$\frac{1}{8} \cdot 4\pi = \frac{\pi}{2}$

15. (7 pts) Find the area of the region enclosed between the parabolas $y = x^2 - x - 1$ and $y = -x^2 - x + 1$.

Solve $x^2 - x - 1 = -x^2 - x + 1$ to see where the parabolas meet : $2(x^2-1) = 0$, $x = \pm 1$. The area is

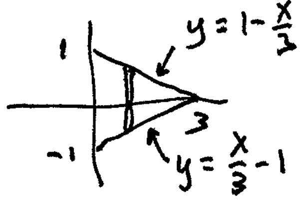
$\int_{-1}^{1} (-x^2-x+1) - (x^2-x-1) \, dx$

$= \int_{-1}^{1} 2 - 2x^2 \, dx = 2x - \frac{2x^3}{3} \Big]_{-1}^{1} = \frac{4}{3} - (-\frac{4}{3}) = 8/3$



16. (7 pts) Find the volume of the solid obtained by rotating the triangle with vertices $(0, 1)$, $(0, -1)$ and $(3, 0)$ about the y axis.

By shells :

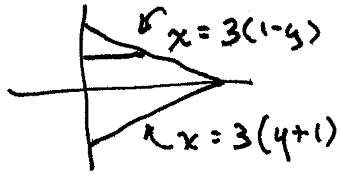


$\int_0^3 2\pi x \left(2 - \frac{2x}{3}\right) dx$

$= 2\pi \left(x^2 - \frac{2x^3}{9}\right)\Big]_0^3$

$= 2\pi(9 - 6) = 6\pi$

By disks :



$\int_{-1}^{0} \pi (3(y+1))^2 \, dy + \int_0^1 \pi (3(1-y))^2 \, dy$

$= 2 \int_0^1 9\pi (1-y)^2 \, dy$ $\quad u = 1-y$, $du = -dy$

$= 18\pi \int_1^0 -u^2 \, du = 18\pi \left(-\frac{u^3}{3}\right)\Big]_1^0 = 18\pi \cdot \frac{1}{3}$

$= 6\pi$

Either method is OK, although shells is easier for this problem