1. [4] Simplify $\cos(\sin^{-1}(2x))$ and state its domain.

Let $u = \sin^{-1}(2x)$, so $2x = \sin u$.

Then $\cos(\sin^{-1} 2x) = \cos u = \sqrt{1 - \sin^2 u} = \sqrt{1-4x^2}$.

The domain is $[-\frac{1}{2}, \frac{1}{2}]$, since the domain of $\sin^{-1}(x)$ is $[-1, 1]$.

2. [4] At which points is the function $f(x) = \begin{cases} 1/(1-e^x) & x < 0 \\ -x & 0 \le x < 1 \\ \cos \pi x & x \ge 1 \end{cases}$ (a) continuous, (b) continuous from the right, (c) continuous from the left, (d) neither?

- continuous from the right at 0.
- continuous everywhere else
  (note $\cos \pi = -1 = \lim_{x \to 1^-} (-x)$, so $f(x)$ is continuous at 1).

3. [4] Differentiate $\ln(e^{\sqrt{2}x} + e^{-\sqrt{2}x})$.

$$\frac{1}{e^{\sqrt{2}x} + e^{-\sqrt{2}x}} \left(\sqrt{2} e^{\sqrt{2}x} - \sqrt{2} e^{-\sqrt{2}x} \right)$$

4. [5] Find the point (a, b) on the graph $y = e^x$ where its tangent line passes through $(0, 0)$.

Slope of tangent line L at $(a, b = e^a)$ is $y'|_{x=a} = e^a$.

For L to pass through $(0,0)$ the slope should equal b/a:

$\frac{e^a}{a} = e^a \Rightarrow e^a = a e^a \Rightarrow a = 1,\ b = e$.

5. [4] Use a linear approximation or differentials to estimate $(8.15)^{2/3}$.

With $y = x^{2/3}$, $x = 8$, $dx = .15$, we get

$y = 8^{2/3} = 4$, $y' = \frac{2}{3} x^{-1/3} = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$, $dy = \frac{1}{3} dx = .05$,

so the estimate is $y + dy = 4.05$.

(The exact value is $8.15^{2/3} = 4.049845\ldots$)

6. [4] Find the limit, either finite or infinite, or explain why it does not exist.

$$\lim_{x\to\pi/2} \frac{e^x - 1}{\cos x}$$

As $x \to \pi/2$, $\cos x \to 0$ and $e^x - 1 \to e^{\pi/2} - 1 > 0$.
Since $\cos x \to 0^+$ for $x \to \pi/2^-$ and $\cos x \to 0^-$ for $x \to \pi/2^+$,
we have $\lim_{x\to\pi/2^-} \frac{e^x-1}{\cos x} = +\infty$, $\lim_{x\to\pi/2^+} \frac{e^x-1}{\cos x} = -\infty$.

Therefore $\lim_{x\to\pi/2} \frac{e^x-1}{\cos x}$ does not exist, even as an infinite limit.

7. [4] Find the limit, either finite or infinite, or explain why it does not exist.

$$\lim_{x\to\infty} \frac{\ln(x+2)}{\ln(x+1)}$$

This has $\frac{\infty}{\infty}$ form, so L'Hospital applies, giving

$$\lim_{x\to\infty} \frac{1/(x+2)}{1/(x+1)} = \lim_{x\to\infty} \frac{x+1}{x+2} = \lim_{x\to\infty} \frac{1+1/x}{1+2/x} = 1.$$

8. [4] Find the limit, either finite or infinite, or explain why it does not exist.

$$\lim_{x\to 1} (x+1)^{x-1}$$

The function $(x+1)^{x-1}$ is continuous for all $x > -1$, so
we can evaluate by direct substitution to get
$(1+1)^0 = 2^0 = 1$.

9. [5] If $x^3 + y^3 = xy + 2$, find dy/dx in terms of x and y.

Differentiate $x^3+y^3 = xy+2$ w.r.t. x to get
$3x^2 + 3y^2 y' = y + xy'$.
Then solve for
$$y' = \frac{3x^2 - y}{x - 3y^2}.$$

10. [5] If $-1 \le f'(x) \le 1$ for all x, and $f(1) = 5$, what can you conclude about the value of $f(4)$?

By MVT, $\frac{f(4)-f(1)}{4-1} = f'(c)$ for some $c \in (1,4)$.

Since $-1 \le f'(c) \le 1$, we get $-3 \le f(4) - f(1) \le 3$. Adding $f(1) = 5$ to all expressions gives $2 \le f(4) \le 8$.

11. [5] Find all local maxima and minima of the function $f(x) = \dfrac{x}{x^2+9}$.

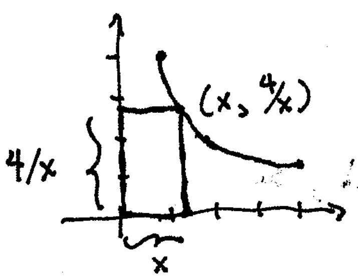
$f'(x) = \dfrac{x^2+9-2x^2}{(x^2+9)^2} = \dfrac{9-x^2}{(x^2+9)^2}$. This has critical points $f'(x) = 0$

at $x = \pm 3$, and sign pattern

$-$		$+$		$-$	f'
decr	-3	incr	3	decr	f

which shows that there is a local min at $x = -3$ and local max at $x = 3$.

12. [5] Find the largest possible perimeter of a rectangle with lower-left corner at $(0,0)$ and upper-right corner on the arc of the curve $xy = 4$ between $(1,4)$ and $(4,1)$.



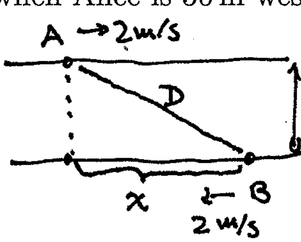
We must maximize the perimeter $2(x + 4/x)$ on $x \in [1,4]$. $P'(x) = 2(1 - 4/x^2)$ has critical point at $x = 2$ ($x = -2$ is not in the domain).

Check perimeter $P(x)$ at crit. point and endpoints:

x	y	$P(x)$
1	4	10
2	2	8
4	1	10

The max perimeter is 10, achieved by rectangles with upper right corner at $(1,4)$ or $(4,1)$.

13. [5] Alice is walking east and Bob is walking west along opposite sides of a street 10 m wide. If each walks at a speed of 2 m/s, how fast is the distance between them decreasing when Alice is 30 m west of Bob?



$D = \sqrt{x^2 + 10^2} = \sqrt{x^2 + 100}$.

We want $-dD/dt$ when $x = 30$ and $dx/dt = -4$.

$\dfrac{dD}{dt} = D'(x)\dfrac{dx}{dt} = \dfrac{x}{\sqrt{x^2+100}}\dfrac{dx}{dt} = \dfrac{30}{\sqrt{1000}}(-4)$,

so $-\dfrac{dD}{dt} = \dfrac{120}{\sqrt{1000}} = \dfrac{12}{\sqrt{10}}$ m/s.

14. [4] For what values of A is the graph of $\cos x + Ax^2$ concave upward at every point?

If $f(x) = \cos x + Ax^2$, then $f''(x) = -\cos x + 2A$.
We have $f''(x) > 0$ for all x when $A > \frac{1}{2}$, since
$-1 \le \cos x \le 1$.

15. [4] Show that $\int (\ln x)^2 \, dx = x(\ln x)^2 - 2x\ln(x) + 2x + C$.

We have to show that d/dx of the right hand side is $(\ln x)^2$.

$$\frac{d}{dx}\left(x(\ln x)^2 - 2x\ln x + 2x + C\right)$$
$$= (\ln x)^2 + x \cdot \frac{2\ln x}{x} - 2\ln x - 2x\frac{1}{x} + 2$$
$$= (\ln x)^2 + \underbrace{2\ln x - 2\ln x}_{0} \underbrace{-2+2}_{0} = (\ln x)^2.$$

16. [5] Evaluate the integral $\int_0^2 |x(x-1)| \, dx$.

Note that $x(x-1) \le 0$ on $[0,1]$, ≥ 0 on $[1,2]$, so $\int_0^2 |x(x-1)| dx$
$$= \int_0^1 -x(x-1)\,dx + \int_1^2 x(x-1)\,dx$$
$$= \int_0^1 x - x^2\,dx + \int_1^2 x^2 - x\,dx = \left.\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + \left.\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = \frac{1}{6} + \frac{2}{3} - \left(-\frac{1}{6}\right) = 1$$

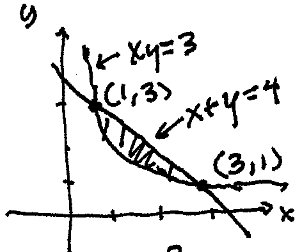
17. [5] Evaluate the indefinite integral $\int \frac{x^3}{x^2+1} \, dx$.

$u = x^2+1 \qquad du = 2x\,dx$
$$\frac{1}{2}\int \frac{u-1}{u}\,du = \frac{1}{2}\int 1 - \frac{1}{u}\,du = \frac{1}{2}(u - \ln u) + C = \frac{1}{2}\left(x^2+1 - \ln(x^2+1)\right) + C'$$

18. [5] Evaluate the integral $\int_0^{\pi/4} \tan x \, dx$. Note $\tan x = \frac{\sin x}{\cos x}$

$u = \cos x \qquad du = -\sin x\,dx \qquad \cos 0 = 1 \qquad \cos \pi/4 = \sqrt{2}/2$
$$-\int_1^{\sqrt{2}/2} \frac{1}{u}\,du = \left.-\ln u\right]_1^{\sqrt{2}/2} = -\ln\frac{\sqrt{2}}{2} = -\ln\frac{1}{\sqrt{2}} = \ln\sqrt{2} = \frac{1}{2}\ln 2.$$

19. [5] Find the area of the region enclosed by the curve $xy = 3$ and the line $x + y = 4$.

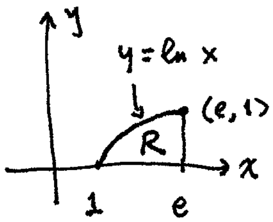


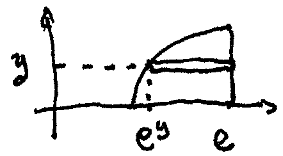
To find endpoints solve $xy=3$ with $x+y=4$, $y=4-x$, $x(4-x)=3$, $x^2-4x+3=0$, $(x-1)(x-3)=0$, $x=1$, or 3. Then

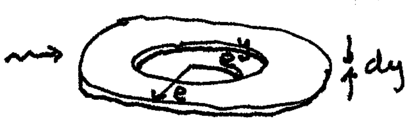
$$A = \int_1^3 4-x-\frac{3}{x}\,dx = 4x-\frac{x^2}{2}-3\ln x\Big]_1^3 = \frac{15}{2}-3\ln 3-\frac{7}{2} = 4-3\ln 3$$

20. [5] Let R be the region bounded by the x-axis, the line $x = e$, and the graph of $y = \ln x$. Set S be the solid obtained by rotating R about the y-axis.

Set up, but do not evaluate, an integral which gives the volume of S using the method of slices.

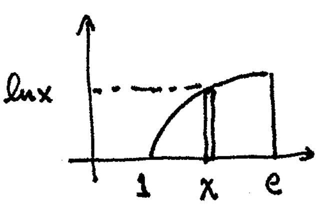


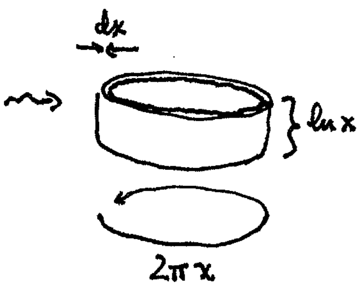




$$V = \int_0^1 \pi e^2 - \pi e^{2y}\,dy$$

21. [5] Set up, but do not evaluate, an integral which gives the volume of the solid S in the previous problem using the method of cylindrical shells.





$$V = \int_1^e 2\pi x \ln x\,dx$$

(The integrals in both these problems evaluate to $\frac{\pi}{2}(1+e^2)$. The one in #20 is easier to do.)

22. [4] Find the average value of $\sqrt{1-x^2}$ on the interval $[-1, 1]$.

$f_{av} = \frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{\pi}{4}$, since the integral is the area of a semicircle of radius 1: