Name ______________________ Section time & instructor ______________________

Student ID______________________

Math 1A—Calculus, Fall 2010—Haiman
Midterm Exam 2

Instructions:

• Write your name, ID number and discussion section time and instructor's name at the top of this page. Do not look at the other pages until the signal to start is given.

• You may use one sheet (written on both sides) of prepared notes. No other notes, books, calculators, computers, cell phones, audio players, or other aids may be used.

• Use your own scratch paper for preliminary work, then write your solutions on the exam paper. Hand in only the exam paper itself.

• Write enough steps or words of explanation so that we can understand how you arrived at your answers. An answer that is just a number or a formula, without any explanation, will not receive partial credit if incorrect, and may not receive full credit even if correct.

• There are 6 questions, 100 total points.

1. (15 pts) Differentiate

$$f(x) = \frac{Ax + B}{Cx + D},$$

where A, B, C, D are constants. Simplify your answer.

By quotient rule,

$$f'(x) = \frac{(Cx+D)A - (Ax+B)C}{(Cx+D)^2}$$

$$= \frac{AD - BC}{(Cx+D)^2}$$

2. (15 pts) Differentiate $x^{(\tan^{-1} x)}$.

Let $y = x^{(\tan^{-1} x)}$

$$\ln y = (\tan^{-1} x)(\ln x)$$

$$y'/y = \frac{\ln x}{x^2+1} + \frac{\tan^{-1} x}{x}$$

$$y' = x^{(\tan^{-1} x)} \left(\frac{\ln x}{x^2+1} + \frac{\tan^{-1} x}{x} \right)$$

3. (15 pts) The pressure, P, and volume, V, of gas in a piston are related by $PV = 500\,\text{N cm}$ (N cm stands for Newton centimeters). What is the rate of change of P when $V = 20\,\text{cm}^3$ and $dV/dt = -5\,\text{cm}^3/s$?

Differentiate $PV = 500$ to get

$$P'V + PV' = 0$$

$$P' = -PV'/V$$

When $V = 20\text{ cm}^3$, $PV = 500\text{ N cm}$ gives

$$P = \frac{500}{20} = 25\ \text{N}/\text{cm}^2$$

Therefore, when $V' = -5\text{ cm}^3/\text{s}$, we get

$$P' = -\frac{25(-5)}{20} = 6.25\ \text{N}/\text{cm}^2/\text{s}$$

4. (15 pts) Find the absolute maximum and minimum values of the function $f(x) = (\ln x)/x$ on the interval $[1, 5]$. You may find it useful to know that $(\ln 5)/5 \approx 0.322$ and $1/e \approx 0.368$.

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

We have a critical number $c = e$, since $1 - \ln e = 0$.
Evaluating $f(x)$ at the endpoints and the critical number,

$f(1) = 0 \leftarrow$ minimum

$f(e) = 1/e \leftarrow$ maximum, since $\frac{1}{e} > \frac{\ln 5}{5}$

$f(5) = \ln 5/5$

5. (a) (10 pts) Find the linear approximation to the function $f(x) = x^3$ at $a = 1$.

$f(a)=1$, $f'(a) = 3x^2\big|_{x=1} = 3$

$L(x) = 3(x-1)+1 = 3x-2$

(b) (5 pts) Find the estimated value of $(1.05)^3$ given by the linear approximation in part (a). Is the estimated value larger than the actual value, or smaller?

$L(1.05) = 1.15$

This is an underestimate, since the slope of the graph of the function, $f'(x) = 3x^2$, is increasing.

6. For the function

$$f(x) = e^{-x^2/2}$$

(a) (10 pts) Find the intervals on which f is increasing or decreasing.

$f'(x) = -xe^{-x^2/2}$

On $(-\infty, 0)$, $f'(x) > 0$, f is increasing.

On $(0, \infty)$, $f'(x) < 0$, f is decreasing.

(b) (10 pts) Find the intervals on which the graph of f in concave upwards or concave downwards.

$f''(x) = (x^2-1)e^{-x^2/2} = (x+1)(x-1)e^{-x^2/2}$

On $(-\infty, -1)$ and $(1, \infty)$, $f''(x) > 0$, concave up.

On $(-1, 1)$, $f''(x) < 0$, concave down.

(c) (5 pts) Find the inflection points on the graph of f.

The concavity changes at $(-1, e^{-1/2})$ and $(1, e^{-1/2})$.