Math 1A | Calculus | Fall, 2006

Prof. Haiman

Practice Exam for Midterm 1—Solutions

1. Find the domain and range of the function

$$f(x) = \frac{1}{(x-2)^2}.$$

The domain is $x \neq 2$. The range is $(0, \infty)$, since $f(x)$ is always positive, approaches 0 for large x, and approaches ∞ for x approaching 2.

2. Express the function

$$u(t) = \frac{\cos t}{1 + \cos t}$$

as a composite $f \circ g$ of two other functions.

$u = f \circ g$ for $f(t) = t/(1+t)$, $g(t) = \cos t$.

3. An exponential function $f(x) = Ca^x$ has $f(1) = 10$ and $f(3) = 40$. Find the constants C and a.

$f(3)/f(1) = a^3/a = a^2 = 4$, so $a = 2$. Plugging in $x = 1$ shows that $C = 5$.

4. Evaluate the limit, if it exists (possibly as an infinite limit).

$$\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3}.$$

$$\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3} = \lim_{x \to 3} \frac{x + 1 - 4}{(x-3)(\sqrt{x+1} + 2)} = \lim_{x \to 3} \frac{1}{\sqrt{x+1} + 2} = 1/4.$$

5. Evaluate the limit, if it exists (possibly as an infinite limit).

$$\lim_{x \to \pi/2} \tan x$$

The limit doesn't exist, not even as an infinite limit, since $\lim_{x \to (\pi/2)^-} \tan x = +\infty$ and $\lim_{x \to (\pi/2)^+} \tan x = -\infty$.

6. Evaluate the limit, if it exists (possibly as an infinite limit).

$$\lim_{x \to \infty} \tan^{-1}(x^2 - x^4)$$

$\lim_{x \to \infty}(x^2 - x^4) = -\infty$, so $\lim_{x \to \infty} \tan^{-1}(x^2 - x^4) = \lim_{x \to -\infty} \tan^{-1}(x) = -\pi/2$.

7. In the definition of the limit

$$\lim_{x \to 1}(5 - 3x) = 2,$$

find a value of δ that works for $\varepsilon = 0.1$.

Any δ less than or equal to $0.1/3$ works. If $|x-1| < 0.1/3$, then $|(5-3x)-2| = |3-3x| = |3x-3| = 3|x-1| < 0.1$.

8. Prove that there is at least one real solution of the equation $xe^x = 1$.

Let $f(x) = xe^x$. It is a continuous function. Observe that $f(0) = 0$, and $f(1) = e$. Since $0 < 1 < e$, the intermediate value theorem shows that $f(c) = 1$ for some $c \in (0,1)$.

9. For what value of the constant c is the function

$$f(x) = \begin{cases} x + c & \text{if } x \leq -1 \\ x^2 - c & \text{if } x > -1 \end{cases}$$

continuous on $(-\infty, \infty)$?

The only possible discontinuity is at $x = -1$. Now, $f(-1) = \lim_{x\to(-1)^-} f(x) = -1 + c$, and $\lim_{x\to(-1)^+} f(x) = 1-c$. To make $f(x)$ continuous at $x = -1$, we must have $-1+c = 1-c$, thus $c = 1$.

10. Differentiate the function

$$f(x) = (\sqrt{x} - 1)(e^x + x)$$

$$f'(x) = (\sqrt{x} - 1)'(e^x + x) + (\sqrt{x} - 1)(e^x + x)' = \frac{1}{2}x^{-1/2}(e^x + x) + (\sqrt{x} - 1)(e^x + 1).$$

11. Differentiate the function

$$f(x) = \frac{1}{x^3 - x + 2}$$

$$f'(x) = \frac{-(x^3 - x + 2)'}{(x^3 - x + 2)^2} = \frac{-3x^2 + 1}{(x^3 - x + 2)^2}.$$

12. Find an equation of the tangent line to the curve

$$y = \sqrt{x}$$

at the point $(4, 2)$.

The slope is $y'(4) = (1/2)4^{-1/2} = 1/4$. The line has an equation of the form $y = (1/4)x + c$. To find c, use $2 = (1/4)4 + c$ at $(4, 2)$, so $c = 1$. Hence the desired equation is $y = x/4 + 1$.

13. Differentiate $e^x(\cos x + \sin x)$.

$$2e^x \cos x$$