Math 1A | Calculus | Fall, 2006

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Practice Final Exam Solutions

1. Differentiate the function

$$y = \frac{(x+1)\sqrt{x+2}}{\sqrt[3]{x+3}}.$$

It's easiest to use logarithmic differentiation, which gives

$$y' = \frac{(x+1)\sqrt{x+2}}{\sqrt[3]{x+3}}\left(\frac{1}{x+1} + \frac{1}{2(x+2)} - \frac{1}{3(x+3)}\right).$$

2. Evaluate the limit if it exists (possibly as an infinite limit).

$$\text{(a)} \lim_{x\to 1^+} \frac{x}{1-x} \qquad \text{(b)} \lim_{x\to 1^-} \frac{x}{1-x} \qquad \text{(c)} \lim_{x\to 1} \frac{x}{1-x}$$

(a) $-\infty$, (b) $+\infty$, (c) doesn't exist.

3. Find all points P on the curve $y = x^2 + 1$ with the property that the tangent line at P passes through the origin.

We have $y' = 2x$. At $P = (c, c^2+1)$, the equation of the tangent line is $y = 2c(x-c)+c^2+1$. It passes through origin if and only if this equation holds at $x = y = 0$, which gives $-c^2+1 = 0$, so $c = \pm 1$. The two points P are $(-1, 2)$ and $(1, 2)$.

4. Use a linear approximation to estimate $\sqrt{37}$.

Let $f(x) = \sqrt{x}$, so $f'(x) = 1/(2\sqrt{x})$. The linear approximation near $x = 36$ is $y = (1/12)(x-36)+6$. With $x = 37$, this gives $\sqrt{37} \approx 6 + 1/12 \approx 6.0833$. For comparison, the actual value to four decimal places is $\sqrt{37} \approx 6.0828$.

5. If $\sin(y-x) = y + x$, express dy/dx in terms of x and y.

$$\begin{aligned}
\cos(y-x)(y'-1) &= y'+1 \\
(\cos(y-x)-1)y' &= \cos(y-x)+1 \\
y' &= \frac{\cos(y-x)+1}{\cos(y-x)-1}.
\end{aligned}$$

6. Find the constant a for which $f(x) = x^3 + ax^2$ has an inflection point at $x = 1$. For this value of a, find the intervals of concavity of $f(x)$.

Compute $f''(x) = 6x + 2a$. We need $f''(1) = 0$, so $a = -3$. From the sign of the second derivative $f''(x) = 6x - 6$, we see that $f(x) = x^3 - 3x^2$ is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.

7. Use Newton's method to find the root of $x^4 + x - 4 = 0$ in the interval $[1, 2]$, correct to 6 decimal places.

The formula for the next approximation is

$$x_{n+1} = x_n - \frac{x_n^4 + x_n - 4}{4x_n^3 + 1}.$$

Starting with $x_1 = 1$ yields $x \approx 1.283782$, accurate to 6 decimal digits at x_5. Starting with $x_1 = 2$, you have to go to x_6 before the result is accurate to 6 digits.

8. Find the points on the parabola $y = x^2$ closest to $(0, 1)$.

We must minimize $d^2 = x^2 + (y-1)^2 = x^2 + (x^2-1)^2$. The derivative is $2x + 4x(x^2-1) = 2x(2x^2 - 1)$, which is zero at $x = 0$, $x = \pm 1/\sqrt{2}$. At $x = 0$, we get $d^2 = 1$, while at $x = \pm 1/\sqrt{2}$ we get $d^2 = 3/4$, so the closest points are $(\pm 1/\sqrt{2}, 1/2)$.

9. Find the limit.

$$\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$$

$$\begin{aligned}
\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) &= \lim_{x \to 1} \frac{x - 1 - \ln x}{(x-1)\ln x} \\
&= \lim_{x \to 1} \frac{1 - 1/x}{\ln x + (x-1)/x} \\
&= \lim_{x \to 1} \frac{x - 1}{x \ln x + x - 1} \\
&= \lim_{x \to 1} \frac{1}{\ln x + 2} \\
&= 1/2,
\end{aligned}$$

where we have used L'Hospital's rule twice.

10. Evaluate the integral.

$$\int_1^2 x\sqrt{x-1}\, dx$$

Substitute $u = x - 1$, $du = dx$ to get

$$\int_0^1 (u+1)\sqrt{u}\, du = \int_0^1 u^{3/2} + u^{1/2}\, du = 2/5 + 2/3 = 16/15.$$

11. Find the area enclosed by the lines $x = 0$, $y = 1$ and the curve $y = \sqrt[3]{x}$.

Integrating with respect to y gives

$$\int_0^1 y^3\, dy = y^4/4\Big]_0^1 = 1/4.$$

Integrating with respect to x gives

$$\int_0^1 (1 - \sqrt[3]{x})\, dx = x - (3/4)x^{4/3}\Big]_0^1 = 1 - 3/4 = 1/4.$$

12. Evaluate the integral.

$$\int_0^{\pi/2} \left|\cos x - \frac{1}{2}\right|\, dx.$$

We have $\cos x - 1/2 \geq 0$ on $[0, \pi/3]$ and $\cos x - 1/2 \leq 0$ on $[\pi/3, \pi/2]$. Our integral is therefore equal to

$$\begin{aligned}\int_0^{\pi/3} \left(\cos x - \frac{1}{2}\right)\, dx + \int_{\pi/3}^{\pi/2} \left(\frac{1}{2} - \cos x\right)\, dx &= \sin x - \frac{x}{2}\Big]_0^{\pi/3} + \frac{x}{2} - \sin x\Big]_{\pi/3}^{\pi/2} \\ &= \sqrt{3}/2 - \pi/6 + \pi/12 + \sqrt{3}/2 - 1 \\ &= \sqrt{3} - \pi/12 - 1.\end{aligned}$$

13. Differentiate the function

$$f(x) = \int_x^{2x} \frac{e^t}{t}\, dt.$$

We have $f(x) = F(2x) - F(x)$, where $F'(x) = e^x/x$ by the Fundamental Theorem of Calculus. Therefore $f'(x) = 2F'(2x) - F'(x) = (e^{2x} - e^x)/x$.

14. Find the most general function $f(x)$ for which $f''(x) = \cos x$.

Antidifferentiating twice gives $f'(x) = \sin x + C_1$, $f(x) = -\cos x + C_1 x + C_2$.

15. Find an interval $[0, c]$ on which the average value of the function $f(x) = x^2 + 2$ is equal to 5.

We require

$$5 = \frac{1}{c}\int_0^c (x^2 + 2)\, dx = \frac{1}{c}(c^3/3 + 2c) = c^2/3 + 2.$$

Then $c^2 = 9$, $c = 3$. We take the positive square root, since we were looking for an interval with left endpoint zero (although the average of $f(x)$ on $[-3, 0]$ is also equal to 5).

16. Set up an integral for the volume of the solid obtained by rotating the region enclosed by the x axis, the line $x = 2$, and the curve $y = \ln x$ about the y axis, using

(a) the method of slices;

(b) the method of cylindrical shells.

Evaluate one of these integrals to find the volume.

Note that $y = \ln x$ crosses the x axis at $x = 1$. The integral for (a) is

$$\int_0^{\ln 2} \pi(4 - e^{2y})\, dy = 4\pi y - \pi e^{2y}/2\Big]_0^{\ln 2} = 4\pi \ln 2 - 2\pi + \pi/2 = 4\pi \ln 2 - 3\pi/2.$$

The integral for (b) is

$$\int_1^2 2\pi x \ln x\, dx.$$

It can be evaluated using integration by parts, a technique you will learn in your next calculus course.

17. Find the volume of a pyramid with a square base of length 2 on each side, and height 3.

Let z be the vertical distance from the point at the top of the pyramid. The horizontal cross-section at z is a square with sides of length $2z/3$ and area $4z^2/9$. The volume is

$$\int_0^3 4z^2/9\, dz = 4z^3/27\Big]_0^3 = 4.$$

18. Evaluate the limit by expressing it as an integral.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i^2}{n^2}.$$

$$\int_0^1 x^2\, dx = 1/3.$$