Math 1A, Calculus **Second Midterm Exam Solutions** Haiman, Fall 2004

1. Find $\frac{d^2}{dx^2}(\sec x)$.

$$\sec^3 x + \sec x \ \tan^2 x$$

2. Differentiate $x^{(e^x)}$.

$$x^{(e^x)} e^x (\ln x + \frac{1}{x})$$

3. If $h(x) = f(g(x))$ and $f(0) = 0$, $g(0) = 1$, $f'(0) = 2$, $g'(0) = 3$, $f'(1) = 4$, $g'(1) = 5$, find $h'(0)$.

$$h'(0) = f'(g(0))g'(0) = f'(1)g'(0) = 4 \cdot 3 = 12$$

4. If $x^2 + y^3 = 17$ and $dx/dt = 10$, find dy/dt when $x = 3$.

Differentiating gives $2x\ dx/dt + 3y^2\ dy/dt = 0$. Solve the original equation for y, getting $y = 2$ at $x = 3$. Then $60 + 12\ dy/dt = 0$, so $dy/dt = -5$.

5. A cube is measured to be 6 cm on each side, with a possible error of $\pm.5$ cm. Use a linear approximation or differentials to estimate the error in computing the volume of the cube.

$V = a^3$, $dV = 3a^2\ da$. With $a = 6$ and $da = .5$, get $dV = 54\,\mathrm{cm}^3$ (giving the answer as $\pm 54\,\mathrm{cm}^3$ is OK too).

6. Find all local and absolute minima and maxima of the function $f(x) = x^2(x + 6)$ on the interval $[-5, 3]$. Include local minima and maxima at the endpoints if there are any.

$f'(x) = 3x^2 + 12x = 3x(x + 4)$ has critical points at $x = -4$, $x = 0$, and sign $f'(x) > 0$ on $[-5, -4)$, $f'(x) < 0$ on $(-4, 0)$, $f'(x) > 0$ on $(0, 3]$. Then $f(-5) = 25$ is a local minimum, $f(-4) = 32$ is a local maximum, $f(0) = 0$ is the absolute minimum, and $f(3) = 81$ is the absolute maximum.

7. Verify that $f(x) = x^3 + x - 1$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$, and find all points c for which the conclusion of the Mean Value Theorem holds.

$f(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, with derivative $f'(x) = 3x^2 + 1$. The MVT guarantees that there is a point c in $(0, 2)$ where $f'(c) = (9 - (-1))/2 = 5$. To find c, solve $3c^2 + 1 = 5$, $c = \pm 2/\sqrt{3}$. The negative solution is outside $(0, 2)$, so the only point is $c = 2/\sqrt{3}$.

8. Compute

$$\lim_{x\to 0} \frac{x + x^2}{e^x - e^{-x}}$$

Using L'Hospital's rule,

$$\lim_{x\to 0} \frac{x + x^2}{e^x - e^{-x}} = \lim_{x\to 0} \frac{1 + 2x}{e^x + e^{-x}} = \frac{1}{2}.$$

9. Use the information below to sketch the graph of $y = (x-1)/x^2$. Show any local or absolute maxima and minima and any inflection points by plotting them on your sketch and labelling them with their x and y coordinates.

- The domain of $f(x) = (x-1)/x^2$ is $(-\infty, 0) \cup (0, \infty)$.
- $\lim_{x \to 0^+} (x-1)/x^2 = \lim_{x \to 0^-} (x-1)/x^2 = -\infty$.
- $\lim_{x \to \infty} (x-1)/x^2 = \lim_{x \to -\infty} (x-1)/x^2 = 0$.
- $y = 0$ at $x = 1$, $y < 0$ on $(-\infty, 0) \cup (0, 1)$, and $y > 0$ on $(1, \infty)$.
- $y' = (2-x)/x^3$; $y' = 0$ at $x = 2$, $y' < 0$ on $(-\infty, 0) \cup (2, \infty)$, and $y' > 0$ on $(0, 2)$.
- $y'' = (2x-6)/x^4$; $y'' = 0$ at $x = 3$, $y'' < 0$ on $(-\infty, 0) \cup (0, 3)$, and $y' > 0$ on $(3, \infty)$.

See answer to Chapter 4.5, Exercise 15 in the textbook.