Notes on partitions and their generating functions

1. Partitions of n.

In these notes we are concerned with partitions of a number n, as opposed to partitions of a set. A partition of n is a combination (unordered, with repetitions allowed) of positive integers, called the *parts*, that add up to n. In other words, a partition is a multiset of positive integers, and it is a partition of n if the sum of the integers in the multiset is n. It is conventional to write the parts of a partition in descending order, for example

$$(7, 5, 2, 2)$$

is a partition of 16 into 4 parts. We write $|\lambda| = n$ to indicate that λ is a partition of n. Some authors also use the notation $\lambda \vdash n$ for this.

We define the following quantities enumerating partitions:

$$p(n, k) = \text{ number of partitions of } n \text{ with } k \text{ parts}$$

$$p(n) = \text{total number of partitions of } n$$

$$q(n, k) = \text{number of partitions of } n \text{ with } k \text{ distinct parts}$$

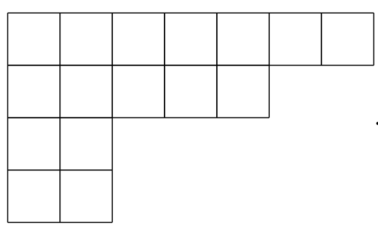
$$q(n) = \text{total number of partitions of } n \text{ with distinct parts}$$

For example, the partitions of 5 are (5), $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$, $(2, 1, 1, 1)$, and $(1, 1, 1, 1, 1)$. Thus $p(5) = 7$, $p(5, 1) = 1$, $p(5, 2) = 2$, $p(5, 3) = 2$, $p(5, 4) = 1$, and $p(5, 5) = 1$, while $q(5) = 3$, $q(5, 1) = 1$, and $q(5, 2) = 2$.

2. Ferrers diagram and conjugate partition

The *Ferrers diagram*, also called *Young diagram*, of a partition $\lambda \vdash n$ is a rectangular array of n boxes, or cells, with one row of length j for each part j of λ.

For example, the diagram of $(7, 5, 2, 2)$ is



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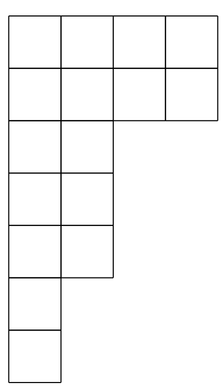
The *conjugate* of a partition $\lambda \vdash n$ is the partition of n whose diagram you get by reflecting the diagram of λ about the diagonal so that rows become columns and columns become rows. We use the notation λ^* for the conjugate of λ. In our example above, with $\lambda = (7, 5, 2, 2)$, the diagram of λ^* is



and so we see that $\lambda^* = (7, 5, 2, 2)^* = (4, 4, 2, 2, 2, 1, 1)$.

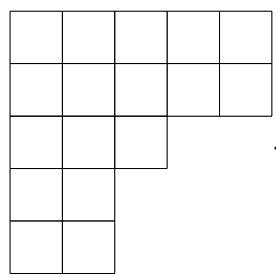
Some obvious but useful facts about λ^* are, first, $(\lambda^*)^* = \lambda$, that is, the conjugate of the conjugate is the original partition and, second, the number of parts of λ is equal to the largest part of λ^*. As a consequence, we see that $p(n, k)$, the number of partitions of n with k parts, is also the number of partitions of n with largest part $\lambda_1 = k$.

The conjugate λ^* can be computed directly without drawing the Ferrers diagrams. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and denote its conjugate by $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$. Notice that $l = \lambda_1^*$ and $m = \lambda_1$. Now λ_i^* is the length of column i in the diagram of λ, and there is one cell in this column for every row of length at least i. Therefore λ_i^* is equal to the number of parts $\geq i$ in λ, or equivalently the largest j such that $\lambda_j \geq i$. For example, we compute

$$(6, 6, 5, 3, 3, 3, 2, 1, 1)^* = (9, 7, 6, 3, 3, 2)$$

because λ_9 is the last 1, λ_7 is the last 2, λ_6 is the last 3, λ_3 is the last part ≥ 4 and also the last 5, and λ_2 is the last 6 in λ.

A partition λ is called *self-conjugate* if $\lambda^* = \lambda$. This means its Ferrers diagram is symmetric, as in for example $\lambda = (5, 5, 3, 2, 2)$:



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We define

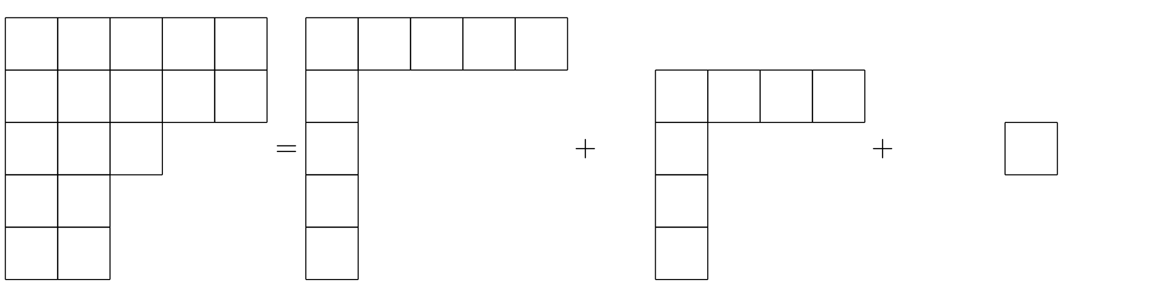
$$k(n) = \text{number of self-conjugate partitions of } n.$$

This number turns up fairly often in partition theory. One interesting fact about $k(n)$ is given by the following theorem:

Theorem: $k(n)$ is also the number of partitions of n into *distinct, odd* parts.

Proof: We'll give a bijection ϕ from $\{\text{self-conjugate } \lambda \vdash n\}$ to $\{\lambda \vdash n \text{ with distinct odd parts}\}$.

Given a self-conjugate λ, define $\phi(\lambda)$ to be the partition whose parts are the "hooks" in the diagram of λ, as illustrated below:



so that $\phi((5, 5, 3, 2, 2)) = (9, 7, 1)$

Each hook has odd size because it is symmetric about the middle, and each hook is strictly larger than the next one, which nests inside it. Therefore $\phi(\lambda)$ has distinct odd parts. On the other hand, given distinct odd numbers, we can form corresponding symmetric hooks and nest them together into a diagram. This operation clearly defines ϕ^{-1} and thus shows we have a bijection.

3. Generating functions for partitions

We begin with the generating function $P(x) = \sum p(n)x^n$ which counts all partitions of all numbers n, with weight x^n for a partition of n.

To choose an arbitrary partition λ of unrestricted n, we can decide independently for each positive integer i how many times to include i as a part of λ.

Each use of i as a part contributes i to the total size n. The generating function for the choice of any number of repetitions of the part i is therefore $1 + x^i + x^{2i} + \cdots = 1/(1 - x^i)$. Multiplying for all i we get

$$P(x) = \sum_n p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i} = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

This infinite product need not disturb us. If we want a particular coefficient $p(n)$ we need only multiply out those factors involving x to a power n or less, and there finitely many of these. Thus the infinite product makes sense since only a finite number of the factors contribute to any given term.

As an exercise to convince yourself this works, you could multiply out the product

$$\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1}{1-x^4}\frac{1}{1-x^5}\cdots,$$

keeping track only of terms up to degree x^5. Then compare the coefficients with the values of $p(0)$ through $p(5)$ which you compute by actually listing all partitions of n for $n = 0$ to 5.

The strategy we used to write down $P(x)$ lends itself to endless variations. Here are some examples.

(1) To count partitions whose parts are $\leq k$, use only the factors for $i = 1, 2, \ldots, k$ to get

$$P_{\leq k}(x) = \sum p_{\leq k}(n)x^n = \prod_{i=1}^{k} \frac{1}{1-x^i} = \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Taking the conjugate partition gives a bijection between partitions of n with parts $\leq k$ and partitions of n with at most k parts. Therefore $P_{\leq k}(x)$ also counts partitions with at most k parts.

(2) To count partitions with exactly k parts, we can again take conjugates and count partitions with largest part equal to k. This is almost the same problem as in (1), except that we should replace the factor $1/(1-x^k)$ with $x^k + x^{2k} + \cdots = x^k/(1-x^k)$ to account for the requirement that we take at least one part equal to k. This gives the generating function

$$P_k(x) = \sum_n p(n,k)x^k = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

(3) To determine the number $o(n)$ of partitions of n with only odd parts, we use only the factors for odd values of i to get the generating function

$$O(x) = \sum_n o(n)x^n = \prod_{i \text{ odd}} \frac{1}{1-x^i} = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}.$$

(4) To count partitions with distinct parts, we must choose for each i whether to use the part i once, or not at all, that is, our partition is a set rather than a multiset. Thus the factor $1/(1-x^i)$ in $P(x)$ must be replaced by $(1 + x^i)$, giving the generating function

$$Q(x) = \sum_n q(n)x^n = \prod_{i=1}^{\infty}(1+x^i) = (1+x)(1+x^2)(1+x^3)\cdots.$$

(5) To count partitions with distinct, odd parts, we combine what we did in examples (3) and (4) to get the generating function

$$K(x) = \sum_n k(n)x^n = \prod_{i \text{ odd}} (1+x^i) = (1+x)(1+x^3)(1+x^5)\cdots.$$

According to the theorem in the previous section, this is also the generating function counting self-conjugate partitions:

$$K(x) = \sum_n k(n)x^n.$$

(6) Another way to get a generating function for $p(n,k)$ is to use a two-variable generating function for all partitions, in which we count each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \vdash n$ with weight $y^k x^n$, where n is the size and k is the number of parts. The monomial giving the weight contribution for a single part of i now becomes yx^i instead of just x^i and accordingly we get the generating function

$$P(x,y) = \sum_{n,k} p(n,k)y^k x^n = \prod_{i=1}^{\infty} \frac{1}{1-yx^i} = \frac{1}{(1-yx)(1-yx^2)(1-yx^3)\cdots}.$$

Setting $y = 1$ gets us back to our original generating function $P(x)$.

4. Partition identities

In the last section we counted $p(n,k)$ in two essentially different ways. One was direct, using the 2-variable generating fucntion

$$P(x,y) = \sum_{n,k} p(n,k)y^k x^n = \prod_{i=1}^{\infty} \frac{1}{1-yx^i}.$$

The other was indirect, using conjugation to get

$$P_k(x) = \sum_n p(n,k)x^n = \frac{x^k}{(1-x)\cdots(1-x^k)}.$$

We can make this last equation into a 2-variable generating function by summing with a factor y^k for all k:

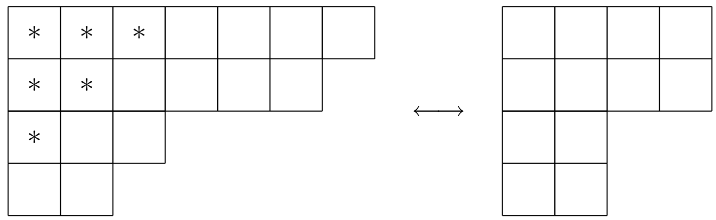
$$P(x,y) = \sum_{n,k} p(n,k)y^k x^n = \sum_k P_k(x)y^k = \sum_k \frac{y^k x^k}{(1-x)\cdots(1-x^k)}.$$

As a result we arrive at a *partition identity*

$$\prod_{i=1}^{\infty} \frac{1}{1-yx^i} = \sum_{k=0}^{\infty} \frac{y^k x^k}{(1-x)\cdots(1-x^k)}$$

expanding an infinite product as an infinite sum of finite products. Though we derived it combinatorially, this is a purely algebraic identity. One theme in combinatorics is to find combinatorial explanations for algebraic identities like this. Often such identities are first discovered in some entirely different context, and are understood combinatorially only later.

We can obtain further identities by more subtle combinatorial analysis of Ferrers diagrams. As an example, we will work out an analog for partitions with distinct parts of what we just did above. Note that if λ has k distinct parts, then its diagram must contain the diagram of the "staircase" partition $(k-1, k-2, \dots, 1)$. Furthermore, it is not hard to see that the rows of the difference are just the parts of an ordinary partition with k parts, and this latter partition can be arbitrary. The figure below illustrates this with $\lambda = (7, 6, 3, 2)$.



The diagram of λ is shown on the left, with the staircase diagram contained in it marked by $*$'s. The corresponding difference partition is shown on the right. To choose a partition with k distinct parts, we can choose an ordinary partition with k parts and then boost it with a staircase. This has the effect of adding $\binom{k}{2}$ to its total size, or multiplying the generating function by $x^{\binom{k}{2}}$. We obtain the generating function

$$Q_k(x) = \sum_n q(n,k)x^n = \frac{x^{k+\binom{k}{2}}}{(1-x)\cdots(1-x^k)}.$$

Now, proceeding just as above but for partitions with distinct parts, we arrive at the partition identity for $Q(x,y) = \sum_{n,k} q(n,k)y^k x^n$:

$$\prod_{i=1}^{\infty}(1+yx^i) = \sum_{k=0}^{\infty} \frac{y^k x^{k+\binom{k}{2}}}{(1-x)\cdots(1-x^k)}$$

The connection between partition combinatorics and algebraic identities can also be applied in reverse, to get surprising combinatorial facts. As an example, consider the generating functions found in the previous section for partitions with distinct parts,

$$Q(x) = (1+x)(1+x^2)(1+x^3)\cdots,$$

and for partitions with odd parts,

$$O(x) = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}.$$

In an effort to simplify $O(x)$, let us write it with a product of $(1-x^i)$ for all i in the denominator, cancelled by the product for even i in the numerator:

$$\begin{aligned} O(x) &= \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} \\ &= \frac{\prod_i (1-x^{2i})}{\prod_i (1-x^i)}. \end{aligned}$$

But $(1-x^{2i})/(1-x^i) = 1 + x^i$, giving

$$O(x) = \prod_i (1+x^i) = Q(x).$$

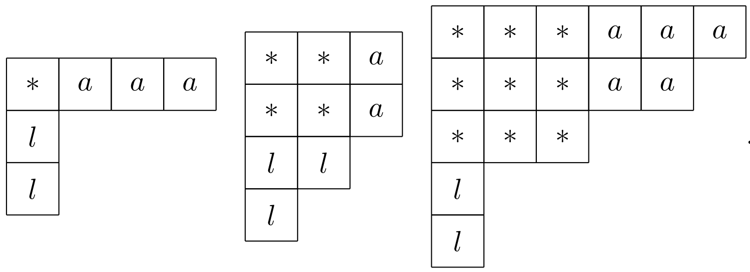
Thus we have proved the following combinatorial identity by simple algebraic manipulation with the generating function.

Theorem: For each n, the number $o(n)$ of partitions of n into odd parts is equal to the number $q(n)$ of partitions of n into distinct parts.

An interesting exercise is to find a bijective proof of the above theorem.

5. The Durfee square

We now consider another way of getting partition identities from analysis of the Ferrers diagrams. Define the *Durfee square* of λ to be the largest square array that fits in the upper left corner of the Ferrers diagram. Here are examples, with the Durfee square marked with $*$'s:



If the Durfee square is c by c, we call c the *Durfee number* of λ. The rest of the diagram of λ consists of two parts, which we call the *arm* (marked above with a's) and the *leg* (marked with l's). The arm and the leg are diagrams themselves. Obviously the arm can be any partition with at most c parts, and the leg any partition with parts at most c.

Suppose we wish to choose a partition (of unrestricted n) with Durfee number c. We can start with the Durfee square and choose the arm and leg independently. The Durfee square contributes c^2 to the total size n. The generating function for the choice of an arm is

$$P_{\leq c}(x) = \frac{1}{(1-x)(1-x^2)\cdots(1-x^c)},$$

and the generating function for the choice of a leg is the same. Multiplying, we find the generating function for partitions with Durfee number c to be

$$x^{c^2} P_{\leq c}(x)^2 = \frac{x^{c^2}}{(1-x)^2(1-x^2)^2\cdots(1-x^c)^2}.$$

Summing for all values of c gives $P(x)$ by the addition principle, leading to the identity

$$\begin{aligned}\prod_{i=1}^{\infty} \frac{1}{1-x^i} &= \sum_{c=0}^{\infty} \frac{x^{c^2}}{(1-x)^2(1-x^2)^2\cdots(1-x^c)^2} \\ &= 1 + \frac{x}{(1-x)^2} + \frac{x^4}{(1-x)^2(1-x^2)^2} + \frac{x^9}{(1-x)^2(1-x^2)^2(1-x^3)^2} + \cdots .\end{aligned}$$

6. Euler's identity

The following wonderful partition identity was discovered by Euler:

$$\prod_{i=1}^{\infty}(1-x^i) = 1 + \sum_{k=1}^{\infty}(-1)^k(x^{(3k^2-k)/2} + x^{(3k^2+k)/2}).$$

This looks a little forbidding but actually the right hand side is very simple. It describes a power series in which most terms are zero and the others have coefficient ± 1; they occur for those exponents x^n where n is of the form $(3k^2 \pm k)/2$. Writing out some terms should make this clearer: the identity reads

$$(1-x)(1-x^2)(1-x^3)\cdots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots .$$

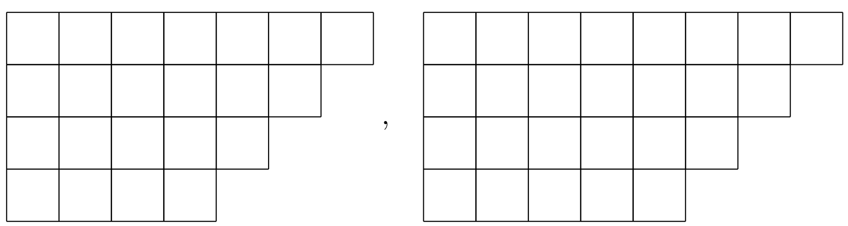
We shall interpret Euler's identity combinatorially and prove it by means of a bijection. By the methods of section 3 we see that

$$Q(x,y) = \prod_{i=1}^{\infty}(1 + yx^i)$$

is the generating function for partitions into distinct parts, counted with weight $y^k x^n$ for a partition λ of n with k parts. We'll use the notation $\lambda \models n$ to mean that λ is a partition of n into distinct parts. Setting $y = -1$ in $Q(x,y)$, each partition into distinct parts will count with weight x^n if it has an even number of parts and $-x^n$ if it has an odd number of parts. Thus the coefficient of x^n in $Q(x,-1)$ is the difference $|\{\lambda \models n, \text{ even \# of parts}\}| - |\{\lambda \models n, \text{ odd \# of parts}\}|$. Now $Q(x,-1)$ is just the left-hand side of Euler's identity, so to prove the identity we must show that the difference $|\{\lambda \models n, \text{ even \# of parts}\}| - |\{\lambda \models n, \text{ odd \# of parts}\}|$ is also counted by the right hand side.

In other words, we are to show that except when $n = (3k^2 \pm k)/2$, there are exactly as many $\lambda \models n$ with an even number of parts as $\lambda \models n$ with an odd number of parts. When $n = (3k^2 \pm k)/2$ we want there to be one extra partition λ with an even number of parts if k is even, and one extra with an odd number of parts if k is odd. In fact, we will see that matters can be arranged so that the extra partition will have k parts. Once we identify the extra partitions, we will then throw them out and find a bijection between all other partitions with an even number of distinct parts, and all other partitions with an odd number of distinct parts.

The extra partitions are going to be the "wedge-shaped" partitions $(2k-1, 2k-2, \ldots, k)$ and $(2k, 2k-1, \ldots, k+1)$, for every k. For $k = 4$ they look like this:



You can easily verfify for yourself that the first wedge is a partition of $(3k^2 - k)/2$ and the second is a partition of $(3k^2 + k)/2$. You can also verify that these wedges are precisely the partitions with the properties (i) all the parts are consecutive, and (ii) the last part is either equal to the number of parts, or one more than the number of parts.

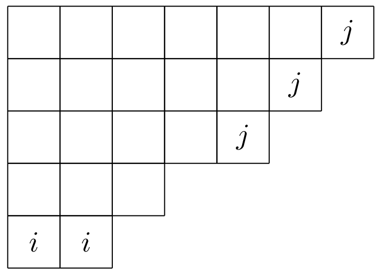
To prove the theorem we now construct a bijection

$$\{\text{non-wedge } \lambda \models n, \ \text{ even \# of parts}\} \leftrightarrow \{\text{non-wedge } \lambda \models n, \ \text{ odd \# of parts}\}.$$

We'll define the bijection by means of an involution S on all non-wedge partitions with distinct parts. An *involution* is an operation such that $S \circ S$ is the identity, *i.e.*, if you do it twice, you get back what you started with. We will arrange for the involution to change the number of parts by 1, that is $S(\lambda)$ will always have exactly one more or one less part than λ. In particular, S sends partitions with an even number of parts to partitions with an odd number, and vice versa. Since S is an involution, it is its own inverse, and thus establishes the required bijection.

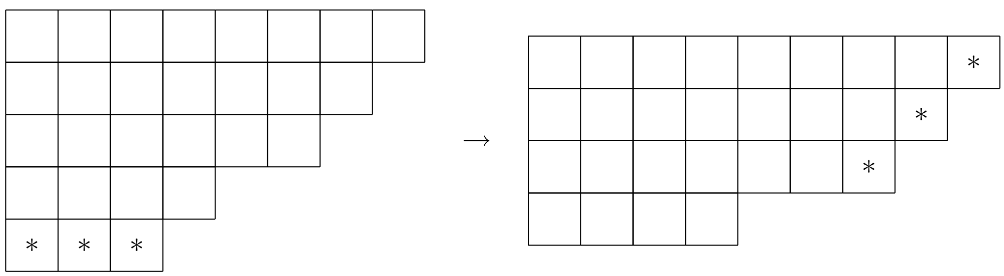
To define S, we use two numbers determined by the diagram of $\lambda \models n$. We define $i(\lambda)$ to be the smallest part of λ. This is the length of the last row in the diagram. We define $j(\lambda)$ to be the number of consecutive parts starting with the largest part. This is the length of the diagonal of cells leading down and to the left from the last cell in the first row of the diagram.

For example, if we take $\lambda = (7, 6, 5, 3, 2)$ then $i(\lambda) = 2$ and $j(\lambda) = 3$ because the first 3 parts $7, 6, 5$ are consecutive. The diagram looks like this:



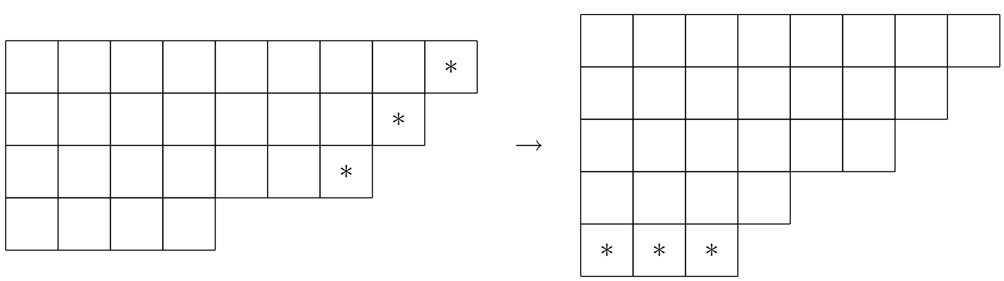
The operation S either transfers the "i" cells up to the "j" position or vice versa, depending which one makes sense. There are two cases:

Case I. $i \leq j$. In this case, remove the last row of λ and extend each of the first i rows by one cell. For example



This makes sense provided the last row (to be removed) is not also among the first i rows (to be extended). But that could only happen if $j \leq i$, hence $j = i$, which would make λ a wedge of the first type, and we excluded those. Notice that after applying S in case I, j for the new partition $S(\lambda)$ is i for the old partition λ. Also i for the new partition $S(\lambda)$ is the length of the (possibly extended) next to last row of λ, so is greater than $i(\lambda)$.

Case II. $i > j$. In this case, remove 1 cell from each of the first j rows and add a new last row of length j. For example,



This makes sense provided the new row of size j that we are trying to add is smaller than the row we are putting it below. The old last row has length i, but after we peel off cells from the first j rows, it might be reduced to $i - 1$, if there were only j rows to begin with. Since we are in the case $i > j$ the only bad possibility is that $i = j + 1$ and the j rows include the last row. But that would mean our partition is a wedge of the second type, which we excluded. Notice that in case II, i for the new partition $S(\lambda)$ is j for the old one λ. Also, after removing 1 from each of the first j rows they are still consecutive, so j for the partition $S(\lambda)$ is at least $j(\lambda)$.

Obviously, S changes the number of parts by 1 in either case. It remains to prove is that S is an involution. In case I, we saw that $j(S(\lambda)) = i(\lambda)$ and $i(S(\lambda)) > i(\lambda)$. Therefore $S(\lambda)$ belongs to case II if λ belongs to case I. Similarly, in case II, we saw that $i(S(\lambda)) = j(\lambda)$ and $j(S(\lambda)) \geq j(\lambda)$, so $S(\lambda)$ belongs to case I if λ belongs to case II. Since the operations in case I and case II undo each other, we conclude that $S(S(\lambda)) = \lambda$ in either case. This completes the proof of Euler's identity.

7. MacMahon's recurrence

Proving Euler's identity is worth the work because it is not only beautiful but also useful. MacMahon observed that it can be used to get an efficient recurrence for the partition numbers $p(n)$. The left hand side of Euler's identity is $1/P(x)$, so we have the generating function identity

$$(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots)P(x) = 1.$$

Considering the coefficient of x^n in this identity, we see that

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \cdots = 0,$$

or

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots .$$

This is of course not an infinite sum; for each n it just continues as long as the terms are $p(j)$ for $j \geq 0$. The starting point is $p(0) = 1$.

How many terms are actually in the recurrence for $p(n)$? Since the terms are $p(j)$ with $j = n - (3k^2 \pm k)/2$, the last term corresponds to the largest k for which $(3k^2 - k)/2 \leq n$. This will be approximately $\sqrt{2n/3}$, so the recurrence involves only about $2\sqrt{2n/3}$ terms. Thus to compute $p(1000)$, say, we need to use only 50 previous values.