Math 172—Combinatorics—Spring 2010
Problem Set 9

Suggested study exercises: Chapter 9, Ex. 1, 3, 7, 8; Chapter 10, Ex. 1, 2, 9.

Problems from the book:

Chapter 9, Ex. 25, 28, 38
Chapter 10, Ex. 21, 34

Additional problems:

A. Recall the formula for Stirling numbers of the second kind that we derived in class using the sieve principle:

$$S(n,k) = \frac{1}{k!}\sum_j (-1)^j \binom{k}{j} (k-j)^n.$$

(i) Show that this formula is equivalent to the generating function identity

$$\sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k.$$

(Eventually we'll see a way to deduce this exponential generating function formula, and therefore also the formula for $S(n,k)$, directly from the definition of Stirling numbers.)

(ii) Use the formula and Problem Set 8, Problem B to prove the algebraic identity

$$\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{j=0}^{k} \frac{(-1)^j/(j!(k-j)!)}{1-(k-j)x}.$$

(Note that the right hand side is an explicit formula for the partial fraction expansion of the left hand side, as you found on Problem Set 8 for $k = 3$, but now for every k.)

B. A *Lisp tree* is an unlabelled rooted tree in which each vertex has at most two children, and these children, if any, are distinguished and labelled as left child and/or right child. If a vertex has only one child we still distinguish between its being a left child or a right child. So, for example, there are 5 lisp trees with three vertices (draw them and check, to be sure you understand the definition).

Let $l(n)$ be the number of Lisp trees with n vertices, and

$$L(x) = \sum_{n=0}^{\infty} l(n)\, x^n$$

the corresponding generating function. Find a closed formula for $L(x)$. Then deduce from it a formula for $l(n)$ in terms of Catalan numbers.

C. A *ternary tree* is a rooted tree in which every vertex has either zero or three children. Let $v(n)$ be the number of ordered, unlabelled ternary trees with n vertices, and

$$V(x) = \sum_{n=0}^{\infty} v(n)\, x^n$$

their generating function. Find an algebraic identity which determines $V(x)$.

D. Same as Problem C, but for *unordered*, unlabelled ternary trees. Note that unordered forests of three trees cannot be counted by a simple application of the product principle, as in Problem C, so here is a hint: consider separately the cases when the forest consists of (a) three identical trees, (b) two identical trees and one different one, or (c) three different trees.