A. (i) By symmetry in \( k \) and \( l \) (conjugating partitions shows
\[ p(n, k, l) = p(n, l, k) \],
the left hand side is equal to
\[ \sum_{\pi \in \mathcal{P}_k n} p(n, k, l) x^n s^k t^l \]. (Actually, I meant this to be the left hand side in the first place, but I didn’t correct this typo since the identity is also valid with it uncorrected.)

Comparing coefficients of \( s^k \), we have only to prove that for given \( k \),
\[ \sum_{\pi \in \mathcal{P}_k n} p(n, k, l) x^n t^l = \frac{x^k t}{(1-xt)(1-x^2t) \cdots (1-x^kt)} \]

Here we are counting partitions with largest part \( k \) by
weights \( n \): size of the partition, \( l \): # of parts. We can independently choose the number of parts equal to \( i \) for each \( i \leq k \).
This contributes a factor \( \frac{1}{1-x^it} \) for each \( i \leq k \), and \( \frac{x^k t}{1-x^kt} \)
for \( i = k \), since we must have at least one part equal to \( k \).
The right hand side is the desired generating function by
the product principle.

(ii) \[ \sum_{\pi \in \mathcal{P}_k n} p(n, k, l) \] is the coefficient of \( s^k t^l \) when we set \( x = 1 \), i.e.,
\[ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{s^k t^l}{(1-t)^k} = t \sum_{l=1}^{\infty} \left( \frac{1}{1-t} \right)^l = t \frac{1}{1-\frac{t}{1-t}} = \frac{st}{1-st} \]
\[ = st \sum_{m=0}^{\infty} (s+t)^m = st \sum_{k \geq 0} \sum_{l \geq 0} (k+l-2) s^k t^l = \sum_{k \geq 0} (k+l-2) s^k t^l \] \[ \text{Binomial Theorem.} \]
Hence \[ \sum_{\pi \in \mathcal{P}_k n} p(n, k, l) = (k+l-2 \choose k-1) = (k+l-2 \choose l-1) \]

(iii) The diagram of a partition with \( l \) parts and largest part \( k \) looks like this:

\[ \begin{array}{c}
\{ \end{array} \]

the boundary is a lattice path with
\( k-1 \) east steps and \( l-1 \) north steps.
There are \( (k-1)!(l-1)! \) \( \left( \frac{(k+l-2)!}{(k-1)!l!} \right) \) ways to choose it.
We deal the last part of the identity before:

\[
\prod_{i \text{ odd}} (1 + x^i) = \prod_{i \text{ odd}} (1 - (-x)^i) = \prod_{i=1}^{\infty} \frac{(1 - (-x)^i)}{(1 - (-x)^i)} = \prod_{i=1}^{\infty} \frac{1}{1 + (-x)^i}
\]

For the first part, \( \prod_{i \text{ odd}} (1 + x^i) = \sum_{n=0}^{\infty} \mathcal{A}(n) x^n \), where \( \mathcal{A}(n) \) is the number of partitions with distinct odd parts, and also of self-conjugate partitions of \( n \). Any self-conjugate partition can be built up as

\[
\lambda = \begin{array}{c}
\text{max} \\
\alpha \\
\alpha'
\end{array}
\]

where the max square is the Durfee square, and \( \alpha \) is a partition with parts \( \leq m \). We have \( |\lambda| = m^2 + 2|\lambda'| \), and the weight enumerator for partitions \( \alpha \) with parts \( \leq m \) is \( \frac{1}{(1-x)(1-x^2) \cdots (1-x^m)} \). Hence the weight enumerator for self-conjugate partitions with Durfee square of size \( m \) is

\[
\frac{x^{m^2}}{(1-x^2)(1-x^4) \cdots (1-x^{2m})}
\]

and summing \( m \) gives the desired identity.

(Note we substituted \( x^2 \) for \( x^2 \) in the weight enumerator the \( \lambda \)'s to account for \( \alpha \)'s of size \( n \) contributing \( 2n \) to the size of \( \lambda \).)
C. (i) Keeping only terms $x^5$ and lower,

\[
\frac{1}{1-x^5} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots
\]

\[
\frac{1}{1-x^2} = 1 + x^2 + x^4 + \ldots \quad \Rightarrow \quad 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + \ldots
\]

\[
\frac{1}{1-x^3} = 1 + x^3 + \ldots \quad \Rightarrow \quad 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \ldots
\]

\[
\frac{1}{1-x^4} = 1 + x^4 + \ldots \quad \Rightarrow \quad 1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + \ldots
\]

\[
\frac{1}{1-x^5} = 1 + x^5 + \ldots \quad \Rightarrow \quad 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \ldots
\]

Hence

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<td>3</td>
<td>5</td>
<td>7</td>
</tr>
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(ii) MacMahon's recurrence gives

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
p(n) & 1 & 2 & 3 & 5 & 7 & 11 & 15 & 22 & 30 & 42 & 56 & 77 & 101 & 135 & 176 & 231 & 297 & 385 & 490 & 627
\end{array}
\]

(iii)

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(sum each row)

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<td>5</td>
<td>7</td>
<td>11</td>
<td>15</td>
<td>22</td>
</tr>
</tbody>
</table>

\[\frac{1}{(1-x)(1-x^2)\cdots (1-x^5)}\]

3. The left hand side is clearly the weight enumerator for partitions with parts of the form $5i+1, 5i+4$, i.e., in the set \[\{1, 4, 6, 9, 11, 14, 16, 19, \ldots \}\].

On the right hand side, to count partitions \(\lambda\) with \(k\) parts \((\lambda_1, \ldots, \lambda_k)\) such that \(\lambda_i \geq 2\lambda_{i+1} + 2\) for \(i=1, \ldots, k-1\), observe that they can also be written \((2\lambda_1, 2\lambda_2, \ldots, 1) + (\mu_1, \ldots, \mu_k)\) where \(\mu_1, \mu_2, \ldots, \mu_k \geq 1\). \(\mu\) is a partition with \(k\) parts. The weight enumerator for such \(\mu\) is

\[\frac{1}{(1-x)(1-x^2)\cdots (1-x^k)}\].

Since \((2\lambda_1, 2\lambda_2, \ldots, 1) + (\mu_1, \ldots, \mu_k)\), the weight enumerator for \(\lambda\) with \(k\) parts is \[\frac{x^{k^2}}{(1-x)(1-x^2)\cdots (1-x^k)}\]. Summing over \(k\) gives the right hand side.