Math 172—Combinatorics—Spring 2010
Problem Set 7

The midterm exam is Friday, March 12 in class. You may bring one (notebook size) sheet of notes, written on both sides. No other notes, books, calculators, computers, or other aids may be used. There will be space on the exam paper to write your answers, but you should bring your own scratch paper.

Homework Problems:

A. Let $p(n,k,l)$ denote the number of partitions of n with l parts, and largest part equal to k. (Since we are specifying a largest part, we'll consider these numbers only for $n > 0$.)

(i) Show that

$$\sum_{n,k,l=1}^{\infty} p(n,k,l)x^n s^l t^k = \sum_{k=1}^{\infty} \frac{x^k s^k t}{(1-xt)(1-x^2t)\cdots(1-x^kt)}.$$

In particular, the expression on the right hand side is symmetric in s and t, which is a rather remarkable algebraic identity.

(ii) Use part (i) to derive a formula for $\sum_n p(n,k,l)$, for given k and l. Note that, if you do this correctly, your result should be symmetric in k and l.

(iii) Find a combinatorial explanation of the result of part (ii).

B. Prove the identity

$$\sum_{m=0}^{\infty} \frac{x^{m^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2m})} = \prod_{i \text{ odd}} (1+x^i) = \prod_{i=1}^{\infty} \frac{1}{1+(-x)^i}.$$

Hint: consider the Durfee square of a self-conjugate partition.

C. (i) Calculate $p(n)$ for $n \le 5$ by multiplying out factors in the generating function

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}.$$

Note that you need only consider the factors for $i = 1, 2, \ldots, 5$, and that you can discard all terms involving x^n for $n > 5$ when multiplying by each factor.

(ii) Caclulate $p(n)$ for $n \le 20$ using MacMahon's recurrence.

(iii) Calculate $p(n,k)$ for $n \le 8$ using the elementary recurrence $p(n,k) = \sum_{m \le k} p(n-k, m)$. Use this to calculate $p(n)$ for $n \le 8$, and check that your results in all three parts of this problem agree.

D. The first *Rogers-Ramanujan identity* is

$$\prod_{i=0}^{\infty} \frac{1}{(1-x^{5i+1})(1-x^{5i+4})} = \sum_{k=0}^{\infty} \frac{x^{k^2}}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Show that this is equivalent to the combinatorial statement that for all n, the number of partitions of n with parts of the form $5i + 1$ or $5i + 4$ is equal to the number of partitions of n all of whose parts differ by at least 2 (in particular, the parts are distinct).

I am not asking you to give a proof these identities, only to show that the algebraic identity is equivalent to the combinatorial one.