Given any permutation \( p = p_1 \cdots p_n \) (in one-line notation), let \( \{ m, m+2, \ldots, m+k \} \) be the set of possible cut points, i.e. the set of numbers \( 0 < m < n \) such that \( p_1 \cdots p_m \) is a permutation of \( \{1, \ldots, m\} \) and \( p_{m+1} \cdots p_n \) is a permutation of \( \{m+1, \ldots, n\} \). Then for each \( i \), \( p_{m+i+1} \cdots p_{m+i+k} \) is a permutation of \( \{m+i+1, m+i+2, \ldots, m+i+k\} \) — this is also valid for \( i = 0 \) and \( k \) if we take \( m_0 = 0, \ m_{k+1} = n \) by definition. Moreover each of these blocks is indecomposable, since a cut point within one of the blocks would also be a cut point for \( p \).

This gives a bijection \( \{ \text{permutations} \} \rightarrow \{ \text{sequences of indecomposable permutations} \} \) such that the length of \( p \) is equal to the sum of the lengths of the indecomposable constituents, i.e. the terms of the sequence \( \mathcal{G}(p) \).

Let \( \mathcal{P} = \{ \text{permutations} \} \), with weight function \( \omega(p) = n \), where \( p \) is a permutation of \( \{1, \ldots, n\} \).

Let \( \mathcal{I} = \{ \text{indecomposable permutations} \} \), with the same weight function. Then if \( \mathcal{f}(n) \) is the number of indecomposable permutations of \( \{1, \ldots, n\} \), with \( \mathcal{f}(0) = 0 \), and we agree not to consider the empty permutation as indecomposable, the weight enumerator of \( \mathcal{I} \) is

\[
\mathcal{F}(x) = \sum_{n=1}^{\infty} \mathcal{f}(n) x^n,
\]

while that of \( \mathcal{P} \) is \( \mathcal{G}(x) = \sum_{n=0}^{\infty} n! x^n \).

By the sequence principle, \( \mathcal{G}(x) = \frac{1}{1 - \mathcal{F}(x)} \), hence

\[
\mathcal{F}(x) = 1 - \frac{1}{\mathcal{G}(x)}.
\]

This can be calculated explicitly: if we set \( \mathcal{H}(x) = 1 - \mathcal{G}(x) \), then \( \mathcal{F}(x) = 1 - \frac{1}{1 - \mathcal{H}(x)} = \mathcal{H}(x) + \mathcal{H}(x)^2 + \cdots \), and taking the first \( n \) powers of \( \mathcal{H}(x) \) in the sum will give the first \( n \) values of \( \mathcal{f}(n) \). I did this on the computer to get for \( n \leq 8 \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \mathcal{f}(n) )</th>
</tr>
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<tbody>
<tr>
<td>0</td>
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<td>8</td>
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</tbody>
</table>

For \( n = 9 \): 29093
The generating function \( \sum_{n=0}^{\infty} p(n, k) x^n t^k = P(x, t) \) is given by \( P(x, t) = \prod_{i=1}^{\infty} \frac{1}{1-x^i t} \) (as we showed in class). Setting \( t = 1 \) gives
\[
\sum_{n=0}^{\infty} (\text{even}(n) - \text{odd}(n)) x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}.
\]

Now, if \( \text{poo}(n) \) denotes the number of partitions of \( n \) with distinct, odd parts, its generating function is given by
\[
P_{\text{odd}}(n) = \sum_{n=0}^{\infty} \text{poo}(n) x^n = \prod_{i \text{ odd}} (1+x^i)
\]

Then \( \text{poo}(n) = \sum_{n=0}^{\infty} (-1)^{n} \text{poo}(n) x^n = \prod_{i \text{ odd}} (1-x^i) \). Note \( (x^n) \) for \( n \), \( x^2 \) for even \( i \), etc.

This shows \( \text{even}(n) - \text{odd}(n) = (-1)^{n} \text{poo}(n) \).

In particular, \( |\text{even}(n) - \text{odd}(n)| = \text{poo}(n) \), and we also get the sign.

A. a) The choice of \( r_i \in \{0, 1, 2, 3, \ldots \} \) gives a factor \( \frac{1}{1-x^i} \),

So \( \sum_{n=0}^{\infty} q(n) x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i} \).

b) Part (a) implies \( q(n) = p(n) \)

c) There is a bijection \( (r_1, r_2, \ldots) \rightarrow (1^{r_1}, 2^{r_2}, \ldots) \)
between compositions of \( n \) with the required divisibility property and partitions of \( n \).

B. A weak composition is a sequence from \( B = \{0, 1, 2, \ldots \} \).

With weight \( w \) for the size of the composition and \( z \) for the number of parts, the weight enumerator for \( B \), which has \( w(r) = r \), \( z(r) = 1 \), is \( t + tx + tx^2 + \cdots = \frac{t}{1-x} \). Then by the sequence principle, the weight enumerator for weak compositions
is given by \( \sum_{x \in A} x \cdot y(x) = \sum_{x < 0} x \cdot e(x) = \frac{1}{1-e(x-x)} = \frac{1-x}{1-x-t} \). Note that this is also equal to

\[
\sum_{n, k=0}^{\infty} \binom{n}{k} x^n y^k = \sum_{n, k=0}^{\infty} \binom{n+k-1}{n} x^n y^k,
\]

since we know that there are \( \binom{n}{k} \) weak compositions of \( n \) with \( k \) parts.

C. a) By the sequence principle,

\[
C(x) = \sum_{n=0}^{\infty} c(n) x^n = \frac{1}{1-A(x)} = \frac{1}{1-x-x^2} = \frac{1-x^2}{1-x-x^2}
\]

\[C(0) = 1 \]
\[C(n) = F_{n-1} \quad \text{for } n > 0\]

D. a) \( (1 + x^{10} + x^{20} + x^{30}) (1 + x^{20} + x^{30} + x^{40} + x^{50}) \) \( (1 + x^5 + x^{10} + x^{15} + x^{20} + x^{25} + x^{30}) \)

b) \( (1 + x^{20})^3 (1 + x^{10})^5 (1 + x^5)^6 \)