5. (24) Imitating the proof of Theorem 5.12, to choose a partition of \([n]\), \([n+1]\) without singletons, if \(i\) is the number of elements not in the block of \(n\), then \(i\) can go from 0 to \(n-1\), and we can choose and partition our \(i\) elements in \(F(i)\) ways, giving

\[
F(n) = \sum_{i=0}^{n-2} \binom{n-1}{i} F(i) \quad \text{for} \quad n > 0 \quad (\text{in particular,} \quad F(1) = 0)
\]

\(F(0) = 1\).

5. (26) We'll prove \(B(n) = F(n) + F(n+1)\) by induction on \(n\). For \(n = 0\), \(B(0) = 1 = F(0) + F(1)\) is correct.

For \(n > 0\), applying the recurrence to the \(F(n+1)\) term gives

\[
P(n) = F(n) + \sum_{i=0}^{n-1} \binom{n-1}{i} F(i) = \sum_{i=0}^{n-1} \binom{n}{i} F(i)
\]

\[
= \sum_{i} \left( \binom{n-1}{i-1} + \binom{n-1}{i} \right) F(i) \quad \text{by Pascal relation}
\]

\[
= \sum_{i} \binom{n-1}{i-1} F(i) + \sum_{i} \binom{n-1}{i} F(i) \quad \text{(change index of summation in first sum)}
\]

\[
= \sum_{i} \binom{n-1}{i} F(i) + \sum_{i} \binom{n-1}{i} F(i) \quad \text{by induction, since only } i \leq n \text{ is involved}
\]

\[
= B(n) \quad \text{by Thm. 5.12}
\]

For a bijective proof, note that \(B(n) - F(n) = \# \text{ partitions of } [n]\) with at least one singleton block. Call the set of these \(G_n\).

We'll find a bijection between \(G_n\) and \(F_{n+1} = \# \text{ partitions of } [n+1]\) with no singletons. Take \(\phi: G_n \rightarrow F_{n+1}\) as follows: given \(\pi \in G_n\), collect all singleton blocks of \(\pi\) together with \(n+1\) to make a block \(B\) of \(\phi(\pi)\), and take the other blocks of \(\phi(\pi)\) to be the non-singleton blocks of \(\pi\). Note that since \(\pi\) has at least one singleton block, \(B\) contains \(n+1\) and at least one other element, so \(B\) is not a singleton, and thus \(\phi(\pi) \in F_{n+1}\) as required.

The inverse map \(\psi: F_{n+1} \rightarrow G_n\) sends \(\tau \in F_{n+1}\) to \(\psi(\tau)\) whose blocks are the non-singleton blocks not containing \(n+1\) in \(\tau\), and singletons.
\[\text{for each } i \neq n+1 \text{ in the block of } n+1 \text{ in } \tau. \text{ Note that there is at least one such } i, \text{ since } \xi_{n+1} \text{ is not a block of } \tau, \text{ so } \psi(\tau) \in \mathcal{G}_n, \text{ as required. Now it's easy to see from the way they are constructed that } \psi(\xi) = \tau \text{ and } \psi(\tau) = \tau, \text{ so we have bijections } \mathcal{G}_n \overset{\psi}{\rightarrow} \mathcal{F}_{n+1}, \text{ hence } B(n) - F(n) = F(n+1).\]

A. our partition has shape

\[
\begin{array}{ccc}
\text{m} & \text{m} & \text{m} \\
\text{a} & & \\
\text{b} & & \\
\end{array}
\]

where \(a\) and \(b\) have \(\xi_n\) parts, and \(|a| + |b| = n\).

With \(|a| = k\), there are \(\sum_{k=0}^{\infty} p_{\leq m}(k)p_{\leq m}(n-k)\) ways to choose \(a\) and \(b\), hence\[
\sum_{k=0}^{\infty} p_{\leq m}(k)p_{\leq m}(n-k)
\]
choices altogether.

To deduce S.(25) take \(m=n, n=2N\). Then \(p_{\leq m}(k)p_{\leq m}(n-k)\) above implies

\[
p(N^2+2N) \geq \sum_{k=0}^{2N} p_{\leq m}(k)p_{\leq m}(2N-k) \geq p_{\leq m}(N)p_{\leq m}(N) = p(N)^2.
\]

B. Claim: \(\lambda\) has \(k\) distinct parts \(\implies\) \(\lambda'\) has exactly the numbers \(1, \ldots, k\) as parts, each at least once.

To see this, observe that \(\lambda\) does not have distinct parts \(\implies\)
the diagram of \(\lambda'\) has two equal columns. But columns \(i\) and \(i+1\) in the diagram are equal \(\implies\) \(\lambda'\) does not have \(i\)
as a part. So \(\lambda'\) has largest part \(k\) (since \(\lambda\) has \(k\)
parts) and \(\lambda'\) omits no part \(\implies\) \(\lambda\) has distinct parts.

C. There is a bijection

\[
\begin{cases}
\lambda \text{ with } k \text{ distinct parts} & \lambda \text{ with } \xi_n \text{ parts, } |\lambda| = n \rightarrow \text{subtract } (k,k-1,\ldots,1) \\
|\lambda| = n - (k+1) \rightarrow \text{add } (k,k-1,\ldots,1) & \xi_n \text{ parts, } |\lambda| = n \rightarrow \end{cases}
\]
D. $(4, 3, 3, 2)' = (4, 4, 3, 1)$

$[10 \text{pts}]$ $(5, 2, 1, 1)' = (4, 2, 1, 1, 1)$

$(9, 5, 4, 3)' = (4, 4, 4, 3, 2, 1, 1, 1, 1)$

The general rule is: the parts of $(\lambda + \mu)'$ are the union (as multisets) of the parts of $\lambda'$ and the parts of $\mu'$.

Proof. A part $i$ occurs $m_i$ times in $\lambda'$ iff $\lambda_i - \lambda_{i+1} = m_i$ (where if $i$ is the length of $\lambda$ we take $\lambda_{i+1}$ to be zero).

This is because $m_i = \# \text{cols of length } i \text{ in diagram of } \lambda = \# \text{cols that reach row } i - \# \text{cols that reach row } i+1 = \lambda_i - \lambda_{i+1}$.

From this recipe for the $m_i$'s it's clear that the multiplicity of $i$ in $(\lambda + \mu)'$, $\lambda_i + m_i - (\lambda_{i+1} - m_{i+1})$, is the sum of the multiplicities in $\lambda$ and in $\mu$ (again using the convention $\lambda_i = 0$ for $i > \text{length}(\lambda)$, when forming $\lambda + \mu$ if $\text{length}(\lambda) < \text{length}(\mu)$).