4.32) A path \((0,0)\) to \((10,10)\) via \((3,3)\) is the same as two paths \((0,0)\rightarrow(3,3)\) and \((3,3)\rightarrow(10,10)\), so there are \(\binom{6}{3}(\binom{14}{7})\) of these. In general, there are \(\binom{k+l}{k}\) paths \((i,j)\rightarrow(i+k, j+l)\).

Now we subtract those which also hit \((5,5)\), which are triple paths \((0,0)\rightarrow(3,3)\rightarrow(5,5)\rightarrow(10,10)\), leaving

\[
\binom{6}{3}(\binom{14}{7}) - \binom{6}{3}(\binom{4}{2})^{10}
\]

4.33) Apply the convolution identity \(\binom{m+n}{k} = \sum_j \binom{m}{j} \binom{n}{k-j}\) with \(m, n, k\) all equal to \(n\), to get

\[
\binom{2n}{n} = \sum_j \binom{n}{j} \binom{n}{n-j} = \sum_j \binom{n}{j}^2 \quad \text{since} \quad \binom{n}{n-j} = \binom{n}{j}
\]

4.35) Expand \(\binom{n}{i}\) using Pascal's relation to get

\[
\sum_{i=0}^{\infty} \binom{n}{i} (-1)^i = \sum_{i=0}^{k} \left(\binom{n-1}{i-1} + \binom{n-1}{i}\right) (-1)^i
\]

\[
= \binom{n-1}{k} - \binom{n-1}{1} + \binom{n-1}{0} - \binom{n-1}{2} + \ldots + (-1)^k \binom{n-1}{k-1} + (-1)^k \binom{n-1}{k}
\]

Then all terms except the last cancel, leaving

\[
\sum_{i=0}^{\infty} \binom{n}{i} (-1)^i = \binom{n-1}{k} (-1)^k.
\]

4.37) First let's integrate both sides of the binomial theorem:

\[
(x+1)^n = \sum_k \binom{n}{k} x^k
\]

\[
\Rightarrow \frac{(x+1)^{n+1}}{n+1} = \sum_k \frac{\binom{n}{k} x^{k+1}}{k+1} + C
\]

Setting \(x=0\) shows that the constant of integration must be

\(C = \frac{1}{n+1}\). Now set \(x=-1\) to get

\[
0 = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{n}{k} (-1)^{k+1} + \frac{1}{n+1},
\]

or equivalently,

\[
\sum_{k=0}^{\infty} \frac{1}{k+1} \binom{n}{k} (-1)^{k+1} = \frac{-1}{n+1}.
\]
Another possible proof: from the formula \( n \choose k = \frac{n^k}{k!} \) we deduce

\[
\frac{n+1}{k+1} \binom{n}{k} = \binom{n+1}{k+1}.
\]

Then

\[
\sum_{k=0}^{n} \frac{n+1}{k+1} \binom{n}{k} (-1)^{k+1} = \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^{k+1} = \sum_{j=1}^{n+1} (-1)^{j} = \left( \sum_{j=0}^{n+1} (-1)^{j} \right) - 1 = (1-1)^{n+1} - 1 = -1
\]

by binomial theorem.

Divide by \( n+1 \) to get

\[
\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} (-1)^{k+1} = \frac{-1}{n+1}.
\]

Note that here we do have to be careful about the range of summation, since \( \binom{n+1}{k+1} \) is not valid for \( k=\text{-}1 \), even though \( \binom{n+1}{k+1} \) is defined and nonzero for \( k=\text{-}1 \).

The general term is

\[
\frac{6!}{m_1! m_2! m_3! m_4!} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}.
\]

To make \( m_1! m_2! m_3! m_4! \) as small as possible with \( m_1 + m_2 + m_3 + m_4 = 6 \).

This is achieved when \( \{m_1, m_2, m_3, m_4\} = \{1, 1, 2, 2\} \), and then the coefficient is

\[
\begin{align*}
(1, 1, 2, 2) &= \frac{6!}{2! 2!} = \frac{720}{4} = 180.
\end{align*}
\]

Note that if any \( m_i \geq 3 \) then \( m_1! m_2! m_3! m_4! \geq 6 \), so \( 2! \cdot 2! = 4 \) is really the smallest possible denominator.

A. Using the binomial theorem with \( \chi = 2 \), \( \psi = 1 \) gives

\[
(2+1)^2 = 3^2 = \sum_{k} \binom{2}{k} 2^k.
\]

For a combinatorial proof, let's interpret the left side as # of words of length \( n \) in symbols \( \{0, 1, 2\} \). To get the right side as a solution to the same counting problem, consider organizing the words according to the value of \( k = \# \text{ of } 1\text{'s and } 2\text{'s in the word. For given } k \text{, there are } n \choose k \text{ choices for which positions are } 0\text{'s and which are } 1\text{'s or } 2\text{'s, then } 2^k \text{ choices for the word of } 1\text{'s and } 2\text{'s in the } k \text{ chosen positions for those symbols, giving a total of } \sum \binom{n}{k} 2^k.

B. The right side, \( \binom{1+2}{k} S(n, 1+2) = \binom{k+2}{k} S(n, k+2) \) counts partitions of \( [n] \) with \( k+2 \) blocks, together with a distinguished subset \( \Psi \) of the blocks. We can count the same thing by first deciding on the value of \( m = \# \text{ of elements in the blocks belonging to } \Psi \), then choosing which \( m \) elements \( X \) these are, \( \binom{n}{m} \) ways, then partitioning \( X \) into our distinguished \( k \) blocks, and \( Y = [n] \setminus X \) into the remaining \( 2 \) blocks, in \( S(m, k) S(n-m, 2) \) ways, for a total of

\[
\sum \binom{n}{m} S(m, k) S(n-m, 2) \text{.}
\]
5(18) "weak" is redundant here since odd parts implies no parts are zero. In particular, subtracting 1 from each of the 5 parts, it is equivalent to counting compositions of 20 into 5 even parts. Now dividing everything by 2, this is the same as counting compositions of 10 into 5 parts, and these are weak compositions — parts = 0 are allowed since they correspond to parts = 1 in the original composition of 25. Finally, the number of these is given by \( \binom{5}{10} = \binom{14}{10} = \binom{14}{4} \).

5(22) A partition of \( \{n\} \) into m 3 blocks must have block sizes either:

\[
\begin{align*}
(4, 1, \ldots, 1) & : \frac{(n + 1)^n}{3^n} = \binom{n}{4} \quad \text{ways} \\
(3, 2, 1, \ldots, 1) & : \frac{(n + 1)^n}{3^n - 3} = \binom{n}{3, 2, n-5} \quad \text{ways} \\
(2, 2, 2, 1, \ldots, 1) & : \frac{(n + 1)^n}{3! (n-6)!} = \binom{n}{2, 2, 2, n-6} / 6 \quad \text{ways}
\end{align*}
\]

Total: \( S(n, n-3) = \binom{n}{4} + \binom{n}{3, 2, n-5} + \binom{n}{2, 2, 2, n-6} / 6 \).