Math 172—Combinatorics—Spring 2010
Problem Set 13

Problems:

A. Given any species $\mathcal{F}$, we can define a new species $\mathcal{T}_{\mathcal{F}}$ such that an element of $\mathcal{T}_{\mathcal{F}}(X)$ consists of a rooted tree T with vertex set X, together with an $\mathcal{F}$ structure on the set of children of each vertex (these $\mathcal{F}$ structures to be chosen independently).

(i) If $F(x)$, $T_F(x)$ denote the exponential generating functions for these species, show that

$$T_F(x) = xF(T_F(x)).$$

For example, when $\mathcal{F}$ is the trivial species, so $F(x) = e^x$, this gives the identity in the notes for the rooted trees generating function $T(x)$.

(ii) Select the species $\mathcal{F}$ so that $\mathcal{T}_{\mathcal{F}}$ is rooted ordered trees, and use (i) to solve for the generating function for these trees. How is it related to the ordinary generating function for unlabelled ordered rooted trees, and why?

(iii) Select the species $\mathcal{F}$ so that $\mathcal{T}_{\mathcal{F}}$ is strictly binary trees (i.e., unordered labelled rooted trees in which every node has either zero or two children), and use (i) to solve for their exponential generating function.

B. Let $\mathcal{T}$ denote the species of rooted trees. Show that its reduced cycle generating function (i.e., with $x = 1$) is given by the equation

$$\widehat{Z}_{\mathcal{T}} = p_1 \cdot (\Omega * \widehat{Z}_{\mathcal{T}}).$$

Here the notation is the same as in Section 5 of the notes on cycle generating functions.

Use this formula to calculate $Z_{\mathcal{T}}(p_1, p_2, \ldots; x)$ up through the x^4 term. For comparison, calculate the same quantity directly by listing the unlabelled rooted trees on up to 4 vertices and describing their automorphism groups (i.e., their stabilizers in the group of permutations of the vertices).

C. Let t_n be the number of unlabelled rooted trees with n vertices, and $T(x) = \sum_n t_n x^n$ the corresponding ordinary generating function.

Show that the formula in Problem B implies the identity

$$T(x) = xe^{T(x)+T(x^2)/2+T(x^3)/3+\cdots}.$$

(You may recall that we derived the same identity in class by formulating a kind of "multiset principle" for ordinary generating functions.)

D. Let f_n denote the number of unlabelled functional digraphs with n vertices, or equivalently, the number of orbits of S_n on the set of functions from $[n]$ to $[n]$ (explicitly, an element $\sigma \in S_n$ acts on a function $h\colon [n] \to [n]$ by $\sigma \cdot h = \sigma \circ h \circ \sigma^{-1}$). Let $F(x) = \sum_n f_n x^n$ be the corresponding ordinary generating function.

Using the formula for the cycle generating function Z_{perm} in the notes, show that

$$F(x) = \prod_{k=1}^{\infty} \frac{1}{1 - T(x^k)},$$

where $T(x)$ is the ordinary generating function for unlabelled rooted trees, as in Problem C.