Math 172—Combinatorics—Spring 2010
Problem Set 12

Suggested study exercises: Chapter 8, Ex. 20, 21

Problems:

A. Call a permutation of a finite set X, together with a linear ordering of its cycles, a *cycle-ordered permutation* of X. Find a formula for the exponential generating function $\sum_{n=0}^{\infty} c_n\, x^n/n!$, where c_n is the number of cycle-ordered permutations of a set X with n elements.

B. An *involution* is a permutation $\sigma\colon X \to X$ such that σ^2 is the identity permutation.

(i) Show that the number of involutions of X is equal to the number of partitions of the set X in which every block has 1 or 2 elements.

(ii) Find a formula for the exponential generating function $\sum_{n=0}^{\infty} t_n\, x^n/n!$, where t_n is the number of involutions of a set X with n elements.

(iii) Derive a formula for t_n from part (ii).

C. Given a combinatorial species $\mathcal{F}$, let $\mathcal{F}_e$ denote the species of $\mathcal{F}$ structures on sets of even size, *i.e.* $\mathcal{F}_e(X) = \mathcal{F}(X)$ if $|X|$ is even, and $\mathcal{F}_e(X) = \emptyset$ if $|X|$ is odd. Similarly, let $\mathcal{F}_o$ be the species of $\mathcal{F}$ structures on sets of odd size.

(i) Show that the exponential generating functions for these species are given by

$$F_e(x) = \frac{F(x) + F(-x)}{2}, \quad F_o(x) = \frac{F(x) - F(-x)}{2}$$

(ii) Use part (i) to show that the exponential generating function for the species of permutations with all cycles of even length is given by

$$\sqrt{\frac{1}{1-x^2}},$$

and for the species of permutations with all cycles of odd length by

$$\sqrt{\frac{1+x}{1-x}}.$$

(iii) Use part (ii) to deduce the result given as Theorem 6.24 in your textbook.

D. A *weighted species* is a species of combinatorial stuctures $\mathcal{F}$, together with a weight function $w\colon \mathcal{F}(X) \to \mathbb{N}$ for every finite set X. As usual for species, we require that the weight assigned to an $\mathcal{F}$ structure on X does not depend on the names of the labels. In particular, the weight enumerator

$$f_n(t) = \sum_{a \in \mathcal{F}(X)} t^{w(a)}$$

should depend only on $n = |X|$, and not on the specific set X chosen. Hence, given a weighted species $(\mathcal{F}, w)$, we can associated to it a mixed ordinary and exponential generating function

$$F(x,t) = \sum_{n=0}^{\infty} f_n(t) \frac{x^n}{n!}.$$

Provided that we define the weight of a product or composite structure to be the sum of the weights of its pieces, the product and composition principles will work for these mixed generating functions just like they do for exponential generating functions (composition means composition as functions of x in this setting).

(i) Consider the species of permutations, weighted by $w(\sigma) =$ (number of cycles of σ), so the mixed generating function is

$$P(x,t) = \sum_{n=0}^{\infty} \sum_{k} c(n,k) t^k \frac{x^n}{n!}.$$

Find a closed formula for $P(x,t)$ from exponential generating function principles.

(ii) Use the (extended) binomial theorem to show that the formula in (i) is equivalent to the formula

$$\sum_{k} c(n,k) t^k = t(t+1) \cdots (t+n-1),$$

given in Lemma 6.13 in your textbook.

(iii) Repeat part (i) for permutations with all cycles of even length.

(iv) Use part (iii) to re-derive the result of Problem Set 8, Problem A.