10.38. This is the special case $k=1$ of Problem D(i), except that we should divide by $n$ to count unrooted spanning trees, giving

$$(n-2)^{n-3}.$$ 

Or, cross out a row and column of $L(A)$ corresponding to one of the endpoints of the missing edge, to get

$$\det \begin{pmatrix} n-2 & -1 \\ -1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ -1 & \cdots & -1 & n-1 \end{pmatrix} = (n-1)^{n-2}$$

and evaluate this determinant by your favorite method.

10.40. (Note that the question in Ex. 38 is answered on the last page of Ch. 10)

(10pts) In each case, calculate subtract the eigenvalues of the adjacency matrix $A(G)$, as given in the solution to Ex. 18, from the regular vertex degree $d$, to get the eigenvalues of the Laplacian $L(G) = dI - A(G)$. Then multiply the non-zero eigenvalues and divide by the number of vertices to get the # of spanning trees:

a) $A$ eigenvalues $\pm 1$  
   $L$ eigenvalues $0, 2$  
   $\#$ spanning trees $2/2 = 1$

b) $2, 0^2, -2$  
   $0, 2^4$  
   $2^2 \cdot 4/4 = 4$

c) $3, 1^3, -1^3, -3$  
   $0, 2^3, 4^3, 6$  
   $2^3 \cdot 4^3 / 6 = 384$

d) $\binom{n-2k}{k}^2 \cdot (2k)^2 \cdot \prod_{k=1}^{n} (2k)^2 / 2^n$

A. To count spanning trees in an undirected simple graph $G$, using the generating function version of the Matrix-Tree Theorem, root the trees at vertex $v_0$, so $\det M_{n,s}(x)$ is the generating function for them, then set the variables to $x_{ij} = 1$ if $i,j \in E(G)$, 0 otherwise, so the generating function reduces to the number of spanning trees with edges in $E(G)$. Then $M_{n,s}(x)$ becomes the matrix $M_0$ in the book's version of the theorem.
B. One method is to use properties of determinants to show that for any matrix $A$, the determinant $\det (A + z_1 \cdots z_n I)$ is a polynomial in the $z_i$'s in which every term has the form $z_{i_1} \cdots z_{i_m}$ for some subset $I = \{i_1, \ldots, i_m\}$ of $\{1, \ldots, n\}$ (this follows from the fact that the determinant is linear in each row) and that the coefficient of $z_{i_1} \cdots z_{i_m}$ is $\det A_{I}$, where $A_I$ is obtained from $A$ by deleting rows and columns $i_1, \ldots, i_m$ (this follows by setting $z_j = 0$ for $j \notin I$, dividing the $i_k$-th row by $z_i$ for $i \in I$, then letting $z_i \to 0$ for $i \notin I$).

Now this lemma applied with $A = M_n(x)$ gives $\det \ell (M_n(x, z)) = \sum_{I} \ell x_{I} \det M_{I}(x) = \sum_{F} x^{F}$.

Another method is to apply the Matrix-Tree theorem for $n+1$, setting $x_{n+1} = z_{n+1}$ and $x_{n+1, j} = 0$ for $j = 1, \ldots, n$. Specializing $M_{n+1}(x)$ this way and then crossing out row and column $n+1$ gives the matrix $M_n(x, z)$, hence $\det M_n(x, z)$ is the generating function for trees on $[n+1]$ rooted at $n+1$, with the variables for an edge $j \to n+1$ set to $z_j$:

```
(0) (0) (0) (0) ...
1 1 1
z_1 z_2 z_3 ...
...
```

But this is the same as the generating function for forests on $[n+1]$ $\sum_{F} x^{F}$.

C. Clarification: what $f_k(G)$ counts will be rooted spanning forests in $G$, that is, each unrooted forest is counted with a factor equal to the product of the sizes of its components, for the number of choices for the roots.

In Problem B, set all $x_{ij} = 1$, all $z_i = z$. Then $M_n(x, z)$ specializes to $L(c) - zI$, so $\det (L(c) - zI)$ is the specialization of $\sum_{F} x^{F}$, i.e. $\sum_{F \in G} (-z)^{|\text{roots}|} = \sum_{F \in G} f_k(G)(z)^{|F|}$. 

(5 pts)
D. Again, to classify, we will count rooted spanning forests in each graph.

i) For $G = K_n^\star 3$, we have

\[
L(G) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-1 & -1 & \cdots & -1
\end{pmatrix} = \lambda I + \begin{pmatrix}
-2 & 0 & \cdots & 0 \\
0 & -2 & \cdots & 0 \\
-1 & -1 & \cdots & -1
\end{pmatrix}
\]

We call this matrix $X$.

Now, the first two rows of $X$ are linearly independent, and their sum is $2(-1, \ldots, -1)$, so all the other rows are in the span of the first two, giving rank $X = 2$, i.e. $X$ has two non-zero eigenvalues.

Since $X(\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}) = (\begin{pmatrix} \frac{1}{2} \\ \vdots \\ 0 \end{pmatrix})$, these eigenvalues are $-n$ and $-2$.

Hence the eigenvalues of $L(G)$ are $0$, $-2$, $n-2$, $n$, giving

\[
\det(L(G) - \lambda I) = (-\lambda)^{-2} (n-2)^2 (n-2)^{n-2} = y (n-2+y)(n+y)^{n-2},
\]

where $y = -2$. By binomial theorem this is equal to

\[
y (n+y)^{n-2} = \sum_{j=0}^{n-2} \binom{n-2}{j} y^j n^{n-2-j},
\]

So $f_\ell(G)$, the coefficient of $y^\ell$ in the above, is given by

\[
\begin{align*}
f_0(G) &= 0 \\
f_1(G) &= (n-2) n^{n-2} \\
f_\ell(G) &= (\frac{n-2}{\ell-2}) n^{n-2} + (n-2)(n-1) n^{n-2-k}, \quad k \geq 2
\end{align*}
\]

ii) For $G = K_{r,s}$, we have

\[
L(G) = \begin{pmatrix}
rI_s & -1 \\
-1 & sI_r
\end{pmatrix}
\]

Vectors of the form \( \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \), $\Sigma a_i = 0$

are eigenvectors with eigenvalue $r$, while those of form

\[
\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \Sigma a_i = 0
\]

have eigenvalue $s$. This gives eigenvalues $0$, $r$, $(r+s-1)$ for $L(G)$, and the missing eigenvalue $\lambda$ must satisfy

\[
0 + (s-1)r + (r-1)s + \lambda = \det(L(G)) = 2rs,
\]

hence $\lambda = r+s$.

Then \( \det(L(G) - \lambda I) = (-\lambda)(r+s-2)(r-2)^{s-1}(s-2)^{r-1} =
\]

\[
y (r+s+y)(r+y)^{s-1}(s+y)^{r-1}, \quad (y = -2)
\]
Applying the binomial theorem to find the coefficient of $y^{k}$, we get

\[ f_{k}(a) = \sum_{i+j=k-2} \binom{s-1}{i} \binom{r-1}{j} r^{s-1-i} s^{r-1-j} + \sum_{i+j=k-1} \binom{s-1}{i} \binom{r-1}{j} r^{s-1-i} s^{r-1-j}. \]