Math 172—Combinatorics—Spring 2010
Problem Set 11

Suggested study exercises: Chapter 10, Ex. 14, 17, 18

Problems from the book:

Chapter 10, Ex. 38, 40

Additional problems:

A. Show that the version of the Matrix-Tree theorem given as Theorem 10.21 in your book is a consequence of the generating function version in the notes.

B. Given a rooted forest F on $[n]$, let's define x^F to be the corresponding monomial in the variables x_{ij} (denoted x_F in the notes), and z^F to be the product of variables z_i for those indices i which are roots of the trees in F. So, e.g., for the forest on page 3 of the notes, we have $z^F = z_2 z_4 z_8$.

Show that the generating function $F_n(\mathbf{x}, \mathbf{z}) = \sum_F x^F z^F$, where the sum is over all forests with vertex set $[n]$, is equal to the determinant of the matrix $M_n(\mathbf{x}, \mathbf{z}) = M_n(\mathbf{x}) + \mathrm{diag}(z_1, \ldots, z_n)$.

C. A *spanning forest* in a graph G is a subgraph with the same vertex set as G, and no cycles (so a connected spanning forest is a spanning tree).

Let $L(G)$ denote the Laplacian matrix of a graph G (undirected, without loops), as in Theorem 10.24. Use the result of Problem B to prove that the characteristic polynomial $\det(L(G) - zI)$ is equal to $F_G(-z)$, where $F_G(z)$ is the generating function $\sum_k f_k(G) z^k$ for the numbers $f_k(G) =$ (number of spanning forests in G with k components).

D. Use the result of Problem C to find formulas for the number of spanning forests of G with k components in the cases:

(i) G is the complete graph K_n with one edge deleted.

(ii) G is the complete bipartite graph $K_{r,s}$.