A. Cayley's formula implies that the number of trees on \([n]\) with degree sequence \(d(1), \ldots, d(n)\) is \(\binom{n-2}{d(1)-1, \ldots, d(n)-1}\). If the \(d(i)\) are positive integers summing to \(n-2\), then this multinomial coefficient is non-zero.

B. i) If \(m\) is the number of degree 3 vertices, and \(k\) is the number (5 each) of degree 1 vertices, then there are \(m+k\) vertices altogether, and the sum of their degrees is \(3m+k = 2(m+k) - 2\), which \(\Rightarrow k = m+2\).

ii) Count the number with degree sequence \(d(1) = \cdots = d(m) = 3,\ d(m+1) = \cdots = d(2m+2) = 1\) then multiply by \(\binom{2m+2}{m}\) choices for the degree 3 vertices, to get

\[
\binom{2m+2}{m} \frac{2m}{2^m} = \binom{2m+2}{m} \frac{(2m)!}{2^m}
\]
C. (i) Our trees look like \[ \text{ordered rooted binary tree, i.e., each vertex has 0 or 2 children. So } A(x) = xB(x) \text{ where } B(x) \text{ is the g.f. for binary trees. Now, the structure of a binary tree is } \]

\[ \text{(or) } \begin{array}{c}
\text{or} \\
\text{(ordered)} \\
\text{or}
\end{array} \]

where the upper part is an ordered pair of binary trees.

\[ \text{hence } B(x) = x + xB(x)^2, \text{ or } xB(x)^2 - B(x) + x = 0. \]

Then \[ B(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \quad \text{(the } + \text{ in } \pm \text{ since the numerator must have constant term 0)}. \]

\[ A(x) = \frac{1 - \sqrt{1 - 4x^2}}{2}. \]

(ii) The Catalan number generating function is

\[ C(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}, \]

so \[ A(x) = x^2 C(x^2) = \sum_{n=0}^{\infty} C_n x^{2n+2}. \] Hence \[ a_{2m+2} = C_m. \]

D. Call the answer to B(ii) \( t_m \). Using the formula \( C_m = \frac{1}{m+1} \binom{2m}{m} \), a little algebra shows that the relationship is

\[ (2m+2)! a_{2m+2} = 2^m (m+2) t_m. \]

This can be explained combinatorially as follows: given an ordered, unlabelled, rooted trivalent tree with root degree 1, the ordering distinguishes all the vertices, so there are \((2m+2)!\) ways to label it. Thus \((2m+2)! a_{2m+2}\) counts labelled ordered rooted trivalent trees with root degree 1. Given a labelled trivalent tree with \(m\) degree 3 vertices and \(m+2\) degree 1 vertices there are \(m+2\) ways to choose a degree 1 vertex as a root, then \(2^m\) ways to order the resulting labelled rooted tree: 2 choices to order the children of each of the \(m\) vertices with 2 children. This gives \(2^m (m+2) t_m\) for the number of labelled ordered rooted trivalent trees with root degree 1.
E. (i) The number of labelled trees on \([n]\) in which 1, \ldots, k are leaves (each) is given by the coefficient of \(x_1 \cdots x_k\) in Cayley's formula, with 
\[x_2x_3 \cdots x_k = x_k = 1,\]
\[x_1 \cdots x_k (x_1 + \cdots + x_k + n-k)^{n-2},\]
which is \((n-k)^{n-2}\). To count trees in which the set of leaves is exactly \([k]\), let \(X\) be the set of those in which 1, \ldots, \(k\) are leaves, and let \(A_i\) be the set of trees in \(X\) in which \(i\) is also a leaf, for \(i = k+1, \ldots, n\). Then \(|A_{x_1} \cap \cdots \cap A_{x_k}| = (n-k-j)^{n-2}\), so the sieve principle gives
\[|X \setminus \bigcap_{i} A_i| = \sum_j (-1)^j \binom{n-k}{j} (n-k-j)^{n-2}.
\]
Now multiply by \(\binom{n}{k}\) choices for which \(k\) vertices should be the leaves.

(ii) Use the formula \(S(n,k) = \frac{1}{k!} \sum_j (-1)^j \binom{k}{j} (k-j)^{n}\) to get
\[\frac{n!}{k!} S(n-2,n-k) = \frac{n!}{k!(n-k)!} \sum_j (-1)^j \binom{n-k}{j} (n-k-j)^{n-2}
= \binom{n}{k} \sum_j (-1)^j \binom{n-k}{j} (n-k-j)^{n-2}.
\]

(iii) Given a tree with \(k\) leaves, labelled 1, \ldots, \(k\), there are \((n-k)!\) ways to label the other \(n-k\) vertices. To get a labelled tree on \([n]\) with \(k\) leaves, labelled 1, \ldots, \(k\), multiplying this by \(\binom{n}{k}\) gives the number of labelled trees on \([n]\) with \(k\) leaves, where the leaves may have any labels. So trees with leaves labelled 1, \ldots, \(k\) and non-leaves unlabelled are counted by \(\frac{n!}{k!} S(n-2,n-k) = S(n-2,n-k)\).