Math 172—Combinatorics—Spring 2010
Problem Set 10

Suggested study exercises: Chapter 10, Ex. 5, 6, 12, 13, 14

Problems from the book:

Chapter 10, Ex. 28 (a good way to organize the counting is to consider all the possible degree sequences, which must be partitions of 12 with 7 parts).

Additional problems:

A. Prove that for every strict composition $(d_1, \ldots, d_n)$ of $2n-2$ into n parts, there exists at least one tree T on $[n]$ such that d_i is the degree of vertex i in T, for all i.

B. A *trivalent tree* is a tree in which every vertex has degree 1 or 3.

(i) Show that every trivalent tree with m vertices of degree 3 has $m+2$ vertices of degree 1. In particular, the total number of vertices $2m+2$ in a trivalent tree is always even.

(ii) Use Cayley's tree enumerator (Notes, Theorem 1) to find an explicit formula for the number of labelled trivalent trees with vertex set $[2m+2]$.

C. Let a_{2m+2} denote the the number of ordered, rooted, unlabelled trivalent trees with $2m+2$ vertices, such that the root has degree one. Let $A(x) = \sum_{m=0}^{\infty} a_{2m+2}x^{2m+2}$ be the corresponding generating function.

(i) Find a closed form formula for $A(x)$.

(ii) Find an explicit formula for a_{2m+2} in terms of Catalan numbers.

D. How are the solutions to part (ii) of problems B and C related, and why?

E. (i) Prove that the number of labelled trees on $[n]$ with k leaves is given (for $n \geq 2$) by the formula

$$\binom{n}{k} \sum_j (-1)^j \binom{n-k}{j} (n-k-j)^{n-2}.$$

Hint: use Cayley's tree enumerator and the sieve principle.

(ii) Show that the number in part (i) is equal to $(n!/k!)S(n-2, n-k)$.

(iii) Deduce that the number of trees with n vertices and k leaves, in which the leaves are labelled with the integers $1, \ldots, k$, and the other vertices are unlabelled, is equal to $S(n-2, n-k)$.